Two-Stage Stochastic Bond Portfolio Optimization Model Using Cash Flow Matching Constraint

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Abstract

This paper introduces the two-stage stochastic model developed for the optimization of bond portfolio. The index of bonds ranges from safe government bonds to risky corporate bonds with different default rates and face values. With the pre-defined initial wealth, the final model constructs a portfolio that suggests how to invest in bonds to meet the liabilities. The final results demonstrate the hedging effects that the stochastic model takes in both first-stage decision and the second-stage decision, by not selecting the highly risky bonds which give most favorable coupon payment. In addition, the comparison between the deterministic model and the stochastic model validates the advantage of using the stochastic model.
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Introduction

1.1 Statement of the Problem

Uncertainties play an important role in financial planning. Looking back at the bond market history, there had been many ups and downs caused by changes in interest rates. There are many factors affecting interest rates: government policy, psychological tenor of people, and business activities. Unfortunately, it is impossible to correctly predict those uncertainties. Thus, the stochastic model is considered as a useful tool in financial planning as opposed to the deterministic model since it helps hedge against worst possible scenarios.

1.2 Purpose of the Project

This project is to develop a two-stage stochastic model that assists in bond portfolio management. At the end, the stochastic model will indicate whether the initial wealth is sufficient to meet the liabilities, along with solutions of both the first-stage decision and the second-stage decision that satisfies the cash flow matching constraints for the given timeline. The portfolio will advise what bonds to purchase, sell, or keep. Furthermore, the developed stochastic model should be proved to be better than the deterministic model.

1.3 Motivation

According to [1], the portfolio created by cash flow matching only consists of risk-free bonds, because risk-free bonds have the default rates small enough to guarantee the coupon rate. Even though those bonds provide the guaranteed returns, there are two problems: 1) Risk-free bonds,
such as government bonds or corporate bonds with supreme credit rating, are expensive. 2) The coupon rate is relatively low compared to more risky bonds.

The two factors that affect the bond price and its return are interest rate and default rates. In the bond market, it is evident that those two factors are closely related. When the economy is prosperous, money circulates well, increasing the interest rate. During the period of prosperous economy, it is easy to find a way to invest the wealth that provides better return than the bonds. As a result, other means of investments are preferred, decreasing the bond trading rate. It should be noted that the chance of default decreases due to the prosperous economy. Therefore, the risk of the purchasing risky bonds, meaning the bonds issued from low credit-ranked firms, decreases.

The opposite scenario happens when the economy is in recession. Due to low interest rates, business activities decreases and many unstable firms default. Therefore, people prefer safe bonds that guarantee the returns even though the return rate is small. Since other means of investment will provide unstable or less interest rates, bonds will be attractive, increasing bond trading rates.

When there is no significant change in the economy, the bonds will return what is expected. This project was initiated from the assertion that the stochastic programming will be able to provide a better portfolio while satisfying the cash flow matching constraints (See Section1.4). The fact that the economy conditions mentioned above will not be known by the time when the decision is made uphold the validity of the stochastic programming.
1.4 Definition of Terms

The key terms used in this paper are as follows:

- **Stochastic model**: Stochastic model considers both volatility as well as variability of random factors which are uncertain when the decision is made, thus providing better solutions that represent real life.

- **Cash Flow Matching**: As one of the many ways to form the bond portfolio, cash flow matching is used to meet the liabilities in the future. It creates a portfolio in such a way that the cash inflow will be exactly same as the cash outflow, which is the liability, at each period.

- **Value of Stochastic Solution (VSS)**: VSS indicates the value of using the stochastic model.

- **Expected Value of Perfect Information (EVPI)**: EVPI represents the price that one would will to pay

- **OPL (Optimization Programming Language)**: It is a programming language developed by ILOG. It provides the necessary data types for optimization model and easy representation of the optimization model.
2 Literature Review

The development of the stochastic bond portfolio optimization model requires an understanding of basic stochastic programming models and characteristic of it. This motivated to study introductory level articles that introduce the use of the stochastic model in general sense. After gaining basic knowledge of the stochastic model, the focus was narrowed down to the financial optimization. It led to study the use of the stochastic model in the financial optimization. Furthermore, the article that introduces cash flow matching was examined. The literatures presented here were either found through the electronic journals or given from the supervisor.

2.1 Benefits of Using Stochastic Model under Uncertainty

In [2], four stochastic programming examples from a variety of areas demonstrate the application of the stochastic model. By providing formulations and the analysis of the result, the author attempts to give general idea of how to formulate uncertainties and proves the benefit of using stochastic programming under the uncertainties.

The author creates simple circumstances but yet those circumstances have uncertainties and multi-decision stages. Throughout the article, the author emphasizes that the stochastic programming is a useful tool to incorporate the uncertainties, which always exist in the real situation. The author admits that the results derived by the perfect information perform the best. In other words, the deterministic programming provides the best objective value if perfect information is given.
But given that the perfect information is not available at the time of making decision, the stochastic programming has advantages over the deterministic programming. Two approaches to prove his assertion are introduced in [2]: expected value of perfect information (EVPI) and value of the stochastic solution (VSS). (See Section 1.4)

The article also mentions that the constraint that enforces nonanticipativity should be inserted. The nonanticipativity constraints enforces that the decision is not based on the knowledge of the future.

2.2 Stochastic Model in the Financial Optimization

According to [2], there are three basic stochastic programming models: 1) anticipative model, 2) adaptive model, and 3) recourse model. The authors emphasize that the recourse model, which is the combination of anticipative model and adaptive model, is commonly used in the financial planning. The recourse model allows making a decision under the uncertainty of the future and then rebalancing the portfolio as a way to adapt to the changes. The authors also provide the mathematical formulations of the three basic stochastic programming models, which help to picture the relationships of the three models.

Furthermore, [2] describes how to formulate the stochastic programming model for the three applications: 1) asset allocation, 2) management of fixed-income securities, 3) asset/ liability management. It is interesting to see that the authors admit that the stochastic programming is one of many possible ways to represent the real-life situation. However, [2] emphasize that when there are uncertainties and the complexity of the scenario trees, the stochastic
programming model is considered as a more practical and proper model compared with other models.

2.3 LP Models with Cash Flow Matching

In [3], Cornuejols introduces linear programming model with cash flow matching constraints. Using a simple example, the author went through the each components of the model. Those components are decision variables, objective, and constraints. The mathematical formulation is later converted to the modeling language. Cornuesjols examines the benefit of using mathematical modeling language.

At the end of the chapter, the author clearly outlines characteristics of the portfolio created by cash flow matching; it is advantageous that the portfolio does not need to be updated because both the cash inflow and the cash outflow are known, but the portfolio is more expensive than the ones that constructed by other methods.
3 Project Goals and Objectives

The volatility and the variability of the finance industry necessitate methods that are able to hedge against potential risks. There is no exception in the bond market. The following sections describe the project goals and objectives that the model needs to achieve in order to incorporate the nature of bond market.

3.1 Primary goal and objectives related

The primary goal of the project is to develop the stochastic bond portfolio optimization model to meet the liability. The model is expected to provide optimal solutions that at least better than the solutions obtained from the deterministic model. The objectives are as follows.

3.1.1 Analyze how the model hedges against the worst scenario

3.1.2 Show that the cost of stochastic model solution is less than the deterministic model solution

3.1.3 Indicate whether the initial wealth is enough to guarantee that the liability at each period is met
3.2 Secondary goal and objectives related

The secondary goal of the project is to develop the model that resembles real life as much as possible. Although some assumptions will be used to minimize the exponential growth of the scenario tree, the constraints will be formulated to reflect the real life as much as possible.

The following describes the objectives related to the secondary goal.

3.2.1 Prove that the stochastic model solution considers all the possible scenarios

3.2.2 Validate that the model reflects the real life by using the original data as much as possible
4 Project Methodology

This chapter describes how the project has been refined throughout project initiation, simple model development, intermediate model development, final model development, data collection, model verification and model validation.

4.1 Project Initiation

At the beginning of the project in September, the scope of the project was narrowed. It was decided that the model should have the following features:

- Stochastic model, incorporating the realistic nature of bond
- Having two-stage decisions
- Having predefined initial wealth
- Providing the bond portfolio consisting of various kinds of bonds, including government bonds and corporate bonds
- Amount of the investment return at each period being at least greater than the liability at each period
- Minimizing the total cost to meet the liabilities

As an optimization language, OPL (See Section 1.4) was chosen because 1) it provides useful data types such as ‘int’ or ‘float’; 2) requires less time to debug; 3) accommodates language context that are equivalent to mathematical function, thus easy to convert from the mathematical function. Furthermore, I had familiarity with OPL because it had been introduced during the
previous course material. In addition, many materials covering sample OPL codes were available through e-library of the University of Toronto.

4.2 Simple Model Development

To enhance the understanding of the stochastic formulation and the ability to write in OPL, the two examples from [2] were used as a practice. One example involved two-stage decision, and the other example involved the financial planning with three-stage decision. After the stochastic model formulation, the result was analyzed to detect the hedge effect that the model takes. Then, it was converted to the deterministic formulation. The numerical values of VSS proved that the stochastic model was more advantageous than the deterministic model.

For the three-decision stage model, the nonanticipativity constraint was written explicitly instead of using succinct form which was introduced in the article. Both examples were relevant to this project, since the scope of the project was two-stage decisions regarding the financial planning.

4.3 Intermediate Model Development

With the knowledge obtained from the literature review and the simple model development, the initial model was developed. Initially, the model was to provide the following information:

- Bond selection for the purchasing at the first-stage decision
• Bond selection for (re)purchasing, selling, or keeping at the second decision stage for rebalancing
• Amount of surplus at the end of each period

The objective function was to minimize the difference between the pre-defined initial wealth and the surplus at the end of the time period. The model was written in a mathematical format first, and then translated into OPL language.

4.4 Data Gathering

Data gathering was as important as developing the model. To run the model, data inputs were required. Those inputs include:

• Liability at each period
• Initial wealth
• Face value, maturity year, coupon rate, default rate of bonds
• Variability of possible bond trading rates
• Probability distribution of bond trading rates

To reflect the reality, the bonds information was gathered from [4]. The total of thirty bonds was collected. The bond selection consisted of ten US government bonds, ten expensive but stable corporate bonds, and ten inexpensive but relatively risky corporate bonds. [4] provided all the information on coupon rate, maturity year, credit ratings, face value, and the yield percentages. The credit ratings were from the three of the NRSRO (Nationally Recognized Statistical Rating...
Organization), which were Moody’s, S&P, and Fitch Ratings. The credit ratings were converted into the default rates of the model in the following manner:

- Range I and default rate of zero, if issued by government
- Range II and default rate of between zero and 10% if the credit ratings are between AAA and BBB
- Range III and default rate of between 10% and 65% if the credit ratings are between BB- and below

4.5 Model Verification and Final Model Modification

Verification process ensured the two criteria: 1) there are no errors or bugs in the model and 2) the algorithm is implemented correctly.

The first criterion was done easily by OPL since OPL identified the lines with error along with the description of the errors. To verify that the second criterion had met, value of the stochastic solution (VSS) was calculated. This method was adopted from [2]. The steps of calculating VSS are as follows:

1) Obtain the solution of the first decision stage under the assumption that the expected trading rate will occur.
2) Insert the solution obtained in 1) into the stochastic model.
3) Calculate the difference between the stochastic model and 2).
If implemented correctly, VSS should reveal gains from using stochastic programming model. The following paragraph explains the rational in calculating VSS.

The objective function is to maximize the surplus. Since the initial wealth is given, the greater the surplus at the end of the timeline, the less did it cost to meet the liabilities. The value obtained from 2) represents the cost when not considering all the possible scenarios in the first stage. Thus, VSS will represent the profit of using the stochastic model, if the algorithm is correct.

However, when the first stage solution from the step 1) was inserted into the stochastic model, OPL indicated that the model was infeasible. The model was diagnosed and a flaw in the logic was found. The first stage solution obtained from the step 1) was not good enough to meet the liabilities for the two out of three possible scenarios before the second decision stages. This verifies the benefits of using the stochastic model solutions.

However, in order to determine the exact gain of using the stochastic model, the objective function was modified to include the penalties of not meeting the liabilities. To differentiate the weights between the surplus and shortage, the penalty of 4 was given to not meeting one dollar of liability. The identical method was used in the financial planning and control example in [2]. At the end, the objective function was changed to maximize the surplus at the end of $t = 6$ minus the sum of penalties (See Section 5.4). As a result, the value of the stochastic solution turned out to be $830.6$ (See Section 6.2).
4.6 Model Validation and the Data Manipulation

Validation ensures that the model correctly represents the behavior of real world from which the model is made. It was expected that the model will construct the bond portfolio which behaves in the following way:

- Include only safe bonds if the trading rate increases
- Include the mix of safe bonds and risky bonds but put more weight on safe bonds, if the trading rate stays the same
- Include more risky bonds when the trading rate decreases

The model, however, behaved in a random manner; for example, the portfolio consisted of mix of safe bonds and risky bonds when the trading rate was set high. It was conclude that the data manipulation was required.

Theoretically, the corporate with higher default rate issues the bonds with the higher coupon rate. For some of bonds, however, the relationship between coupon rates and default rates deviated from the theoretical relationship. The problematic bond data had lower default rate with higher coupon rate – even higher than the risky bonds- or higher default rate with lower coupon rate. Those data were manipulated to get correct representation of bond market. The finalized manipulated data is represented in Table 4.6.1.
After the data manipulation, the model constructed the bond portfolio in the expected way. The coupon rate vs. default rate and face value vs. default rate relationship graphs are shown in Figure 4.6.1.1 and 4.6.1.2.
5 Final Model Descriptions

The model had been continuously refined throughout the project methodology- from the project initiation to the model validation. The following sections will provide detail explanation of the final model. All the formulations introduced here are in mathematical format.

5.1 Assumptions

Assumptions are made to simplify the model development and the data analysis. The following assumptions are made in this paper.

- It is assumed that the total time period $t$ of model is six years and that there are only two decision stages; one at $t = 0$, and the other at the end of $t = 3$. At the first decision stage, the bond portfolio is formed. At the second decision stage, the bond portfolio is rebalanced. The pictorial description is shown in Figure 5.1.1
- The coupon rate is paid on yearly base.
- The surplus is not invested, but carried over to the next period
- Each dollar of shortage has a penalty of 4
- The shortage in the previous period cannot be paid off at later time
- All transaction cost is ignored.
- The credit-ranking stays the same during the six-year period.
- At each decision stage, there exist three possible scenarios– increase of the interest rate, no changes in the interest rate, and decrease of the interest rate. Since the model has two
decision stages, there is the total of nine possible scenarios at the end of period six. The scenario tree is shown in Figure 5.1.2

- The scenario can change only at the time of rebalancing.
- The penalty of the default rate changes depending on the interest rate.
- The probability of all three scenarios is same.

5.2 Model Inputs

To run the OPL model, data file needed to be constructed. The data file includes:

- $F_i$ : Face value of bond $i$ ($i=1, 2, \ldots, 30$)
- $C_i$ : Coupon rate of bond $i$
- $M_i$ : Maturity year of bond $i$
- $D_i$ : Default rate of bond $i$
- $L_t$ : Liability of period $t$
- $T_s$ : Bond trading rate at scenario $s$
- $P_s$ : Probability of scenario $s$
- $W$ : Initial wealth

Part of $F_i$, $C_i$, $M_i$, $D_i$ has the manipulated bonds’ information from the validation stage was used. For the liability $L_t$, numbers were made up. The probabilities of scenario $s$, $P_s$, are set the same for the sake of simplicity. Also, $T_s$ are set at above average (+ 30%), average, and below average (- 30%). The average is set at $T = 1.0$, meaning no change in the bond market.
Initially, \( W \) is set in any number. However, if the number is set too small, the objective value will be negative. Even if the objective value is positive, decision variables on shortage will indicate that the initial wealth has to increase in order to meet cash flow constraints at every period. It is recommended to set \( W \) a little below from the sum of the total liability, then rearrange it accordingly depending on the value of the objective function. (See Appendix A for the data file code)

5.3 Decision Variables

The model has ten decision variables in total. Table 5.3.1 shows what decision variables are decided at which point in the timeline. The first decision stage, which is at time 0, decides the following variables. It should be noted that the values of these two variables are identical for all the possible scenarios.

\[
X_i = \text{Amount of bond } i \text{ purchased at the first decision stage}
\]
\[
Z_0 = \text{Amount of surplus at the end of period } t = 0
\]

Depending on \( X_i \), the surplus varies. Between the first decision stage and the second decision stage, the liabilities of \( t = 1, t = 2, \) and \( t = 3 \) will be paid off, and there could be surplus or shortage after paying the liabilities or shortages, which are defined by,

\[
ZZ_{ts} = \text{Amount of surplus at period } t \text{ when scenario } s \text{ was taken } (t = 1, 2, 3)
\]
\[
QQ_{ts} = \text{Amount of shortage at period } t \text{ when scenario } s \text{ was taken } (t = 1, 2, 3)
\]
Since it was assumed that the market has three possible scenarios at each decision stage, the surplus or shortage will be different depending on the scenario it takes. After paying off the liability at $t = 3$, the portfolio is rebalanced before $t = 4$. The rebalancing will be depending on the past history, which are the histories of the first-periods ($t = 1, 2, 3$). The following decision variables are set during this rebalancing period.

\[ Y_i = \text{Amount of bond } i \text{ purchased at the first decision stage when the past scenario was } s \]

\[ S_i = \text{Amount of bond } i \text{ sold at the first decision stage when the past scenario was } s \]

\[ K_{is} = \text{Amount of bond } i \text{ kept at the first decision stage when the past scenario was } s \]

\[ Z_{3s} = \text{Amount of surplus after rebalancing before period 4 when scenario } s \text{ was taken} \]

The model evaluates the portfolio performance and adapts the model accordingly. The first three decision variables suggest which bonds to (re)purchase, sell, and keep. The amount of surplus after this rebalancing will be different from the past histories, resulting in $ZZZ_{31}$, $ZZZ_{32}$ and $ZZZ_{33}$. After this rebalancing period, which is the second stage decision, the rebalanced portfolio will be carried over afterwards. Since there are also three possible scenario branches, the rebalanced portfolio will perform differently, resulting in nine different surplus (or shortages) per each scenario. The surplus and the shortage between $t = 4$ and $t = 6$ are represented as follows.

\[ Z_{sk} = \text{Amount of surplus at period } t \text{ when scenario } k \text{ was taken given that the past history is scenario } s \]

\[ Q_{sk} = \text{Amount of surplus at the end of period } t \text{ when scenario } k \text{ was taken given that the past history is scenario } s \]
5.4 Objective Function

The model has the following objective function. Entire OPL code is shown in Appendix B.

\[
\text{Maximize } \sum_{i=1}^{3} \sum_{j=1}^{3} (Z_{6k}^{s} P_{k} P_{j}) - 4(\sum_{i=1}^{3} \sum_{j=1}^{3} Q_{i} Q_{j} P_{s}) + 6(\sum_{i=4}^{3} \sum_{j=1}^{3} Q_{i} Q_{j} Q_{k} P_{s})
\]

Given the pre-defined initial wealth, the amount of surplus at \( t = 6 \) implies how well the portfolio has performed during the timeline. Thus, the maximization of \( ZZ_{6k}^{s} \) for all \( k \) and \( s \), guarantees that the model adjusts its performance properly in the second decision stage given the past scenario \( s \).

Depending on how the initial wealth is defined and how well a portfolio is constructed, a shortage could occur. With the first-stage decisions, \( QQ_{s} \) are decided. With the decisions made during the rebalancing, \( QQQQ_{6k}^{s} \) are decided. Thus the shortage implies how well the portfolio has performed before and after the rebalancing.

Here are a few points made to distinguish having surplus and shortage. First, the shortages are counted at every period, whereas the surplus counts only at the end of \( t = 6 \). As mentioned in the assumptions (Section 5.1), the surplus are accumulated and carried over to the following periods. However, the shortage cannot be accumulated and paid at later periods. In other words, the liability must be paid off in the specified period. Thus, the shortage at each period needs to be treated individually.
Second point is that every dollar of shortage occurs causes a penalty of 4, which is adopted from [2]. This point relates to the previous point that the shortage cannot be paid at later times. In other words, shortages need to be borrowed to pay the liability. Since borrowing causes a certain interest rate that is higher than the return on investment and also other disadvantages such as additional paper works, or having a sense of instability, a dollar of shortage is assumed to have four dollars of loss. The concave utility function derived from this setting is shown in Figure 5.4.

5.5 Constraints

The constraints can be sub-divided into three categories, and this section will examine the constraints in the following order.

1) Limit on initial bond purchase
2) Cash flow matching constraint
3) Rebalancing constraint

5.5.1 Limit on initial bond purchase
The constraint (5.5.1.1) guarantees that the amount of bonds purchased at \( t = 0 \) costs below the initial wealth. The decision variable \( Z_0 \) represents the amount of initial wealth left after purchasing the bonds.
\[
\sum_{i=1}^{n} F_i X_i + Z_0 = W \quad (5.5.1.1)
\]

The surplus \( Z_0 \) is then changed to as follows.

\[
Z_0 = ZZ_{01} = ZZ_{02} = ZZ_{03} \quad (5.5.1.2)
\]

The constraint (5.5.1.2) is necessary for the cash flow matching constraints which will be introduced at the next section. It explicitly shows that the surplus at \( t = 0 \) is equivalent for \( s = 1, 2, 3 \).

### 5.5.2 Cash flow matching constraint

The cash flow matching constraint is a critical constraint in this model as it helps satisfying the project goal. They are slightly different before and after the rebalancing point. However, the default penalty is applied in the same manner.

#### 5.5.2.1 First-periods: \( t = 1, 2, 3 \)

The three constraints below guarantees that the cash inflow will offset the liability at \( t = 1, 2, 3 \). If there is a shortage in any scenario \( (s = 1, 2, 3) \), it will be represented as \( QQ_{ts} \).

\[
\sum_{i \in M_{t} > t} \left\{(1 - D \ i \ F_{i} X_{i}\right\} \left(1 - D \ i \right) F_{i} X_{i}\right\} + \sum_{i \in M_{t} = t} \left\{(1 - D \ i \ ) F_{i} X_{i}\right\} + ZZ_{(i-1)s} ZZ_{ts} + QQ_{ts} = L_t \quad \text{for } t = 1, 2, 3 \text{ AND } s = 1 \quad (5.5.2.1.1)
\]

\[
\sum_{i \in M_{t} > t} \left\{(1 - D^2 \ i \ ) C_{i} F_{i} X_{i}\right\} + \sum_{i \in M_{t} = t} \left\{(1 - D^2 \ i \ ) F_{i} X_{i}\right\} + ZZ_{(i-1)s} ZZ_{ts} + QQ_{ts} = L_t \quad \text{for } t = 1, 2, 3 \text{ AND } s = 2 \quad (5.5.2.1.2)
\]
\[
\sum_{i \in M_i > t} \left\{ (1-D^3_i)C_iF_iX_i \right\} + \sum_{i \in M_i = t} \left\{ (1-D^3_i)F_iX_i \right\} + ZZ_{(t-1),t} - ZZ_{ts} + QQ_{ts} = L_t
\]

for \( t = 1, 2, 3 \) AND \( s = 3 \) \hspace{1cm} (5.5.2.1.3)

The constraint (5.5.2.1.1) defines the case of \( s = 1 \), meaning when the scenario tree is taken a path of \( T = 1.3 \). High bond trading rate signifies that bonds are preferred to other means of investments. In most cases, this effect is caused by economy recession. During this period, interest rates decreases, causing unstable businesses tend to default. Thus, the penalty for default rate in the constraint (5.5.2.1.1) is set as \( (1 - D_i) \), the worst of all three scenarios.

The first summation in (5.5.2.1.1) denotes that the coupon rate is paid for the bonds that have the maturity greater than the period \( t \). If the bond is at maturity, the second summation applies, and the face value gets paid. In both summation, the default penalty of \( (1 - D_i) \) applies, meaning that only \( (1 - D_i) \% \) are paid. Also, the surplus from the previous periods, \( ZZ_{(t-1),t} \), is carried over to the next periods. At \( t = 1 \), the previous surplus \( ZZ_{01} \), which is set equal to \( Z_0 \), will be used.

The constraint (5.5.2.1.2) is used when the scenario tree is taken a path of \( T = 1.0 \). In this case, the penalty for default is reduced and set moderate as \( (1 - D_i^2) \). Again, at \( t = 1 \), the previous surplus \( ZZ_{02} \) will be carried over and used.

The constraint (5.5.2.1.3) is used when the scenario tree is taken a path of \( T = 0.7 \). Low bond trading rate indicates that the interest rate has increased, bring people to other investment
market such as stocks. However, due to the high interest rate, it is less likely that the unstable company becomes default during this period. Thus, the penalty for default is set to \((1 - D^3_i)\), which is the minimum of all three cases.

5.5.2.2 Second-periods \((t = 4, 5, 6)\)

The three constraints below guarantees that the cash inflow will offset the liability at \(t = 4, 5, 6\). If there is a shortage in any scenario \((s = 1, 2, 3)\), it will be represented as \(QQQQ_{skx}\).

\[
\begin{align*}
\sum_{i \in M_i, t > t} \left\{ (1 - D^1_i) C_i F_i K_{is} \right\} + \sum_{i \in M_i, t > t-3} \left\{ (1 - D^1_i) C_i F_i Y_{is} \right\} + \sum_{i \in M_i, t = t} \left\{ (1 - D^1_i) F_i K_{is} \right\} \\
+ \sum_{i \in M_i, t = t-3} \left\{ (1 - D^1_i) F_i Y_{is} \right\} + ZZZZ_{(t-1)ks} - ZZZZ_{skx} + QQQQ_{skx} = L_t 
\end{align*}
\]

for \(t = 4, 5, 6\) AND \(s = 1\) \((5.5.2.2.1)\)

\[
\begin{align*}
\sum_{i \in M_i, t > t} \left\{ (1 - D^2_i) C_i F_i K_{is} \right\} + \sum_{i \in M_i, t > t-3} \left\{ (1 - D^2_i) C_i F_i Y_{is} \right\} + \sum_{i \in M_i, t = t} \left\{ (1 - D^2_i) F_i K_{is} \right\} \\
+ \sum_{i \in M_i, t = t-3} \left\{ (1 - D^2_i) F_i Y_{is} \right\} + ZZZZ_{(t-1)ks} - ZZZZ_{skx} + QQQQ_{skx} = L_t 
\end{align*}
\]

for \(t = 4, 5, 6\) AND \(s = 2\) \((5.5.2.2.2)\)

\[
\begin{align*}
\sum_{i \in M_i, t > t} \left\{ (1 - D^3_i) C_i F_i K_{is} \right\} + \sum_{i \in M_i, t > t-3} \left\{ (1 - D^3_i) C_i F_i Y_{is} \right\} + \sum_{i \in M_i, t = t} \left\{ (1 - D^3_i) F_i K_{is} \right\} \\
+ \sum_{i \in M_i, t = t-3} \left\{ (1 - D^3_i) F_i Y_{is} \right\} + ZZZZ_{(t-1)ks} - ZZZZ_{skx} + QQQQ_{skx} = L_t 
\end{align*}
\]

for \(t = 4, 5, 6\) AND \(s = 3\) \((5.5.2.2.3)\)
The coupon payment, bonds payment at maturity, penalty related to the default rate, and shortage are identical to the first-periods (See Section 5.5.2.1). The only difference is that the bonds that are purchased at the rebalancing period are added.

5.5.3 Rebalancing Constraint

After paying the liability at \( t = 3 \), the portfolio is rebalanced. The adjustment will be different depending on the previous path it has taken. Through the adjustment, it is decided that whether to sell or keep the bonds that were purchased at \( t = 0 \) and had maturity year of greater than 3. Also, additional bonds are purchased.

There are three different bond trading rates. As explained in Section 5.5.2.1, the trading rate is affected by the interest rate. The trading rate will be decided by the scenario branch it has taken up to \( t = 3 \). In other words, if \( s = 1 \) has been taken for first-periods, \( T = 1.3 \) will be used as the trading rate. At the end of the rebalancing, the new surplus, \( ZZZ_{3s} \), will be defined. The constraint (5.5.3.1) is used in rebalancing.

\[
ZZZ_{3s} + \sum_{i \in M \mid i > 3} (F_i S_i T_i) - \sum_{i=1}^{n} (F_i Y_i T_i) = ZZZ_{3s} 
\]

for all \( s \)  

(5.5.3.1)

\[
X_i - S_{is} = K_{is} \quad \text{for} \quad i \in M \mid i > 3 \quad (5.5.3.2)
\]

\[
ZZZ_{3s} = ZZZZ_{31s} = ZZZZ_{32s} = ZZZZ_{33s} \quad \text{for all} \quad s \quad (5.5.3.3)
\]
The constraint (5.5.3.2) makes certain that the bonds that are not sold will be kept. Again, this applies only for the bonds that have maturity year greater than 3. The constraint (5.5.3.3) is inserted for the cash flow matching constraints used at t = 4, 5, 6. Each of three $ZZ_1$, $ZZ_2$, and $ZZ_3$ will branch into three bond trading rate, thus making nine different scenarios at the end of t = 6. This circumstance requires a new set of variables for t = 4, 5, 6, which is $ZZZ_{tks}$. The decision variable $ZZZ_{tks}$ takes care of both scenario branches in the first-periods and the second-periods.
6 Discussion and Result Analysis

This chapter outlines the outcome of the stochastic model and discusses the advantages that the stochastic model has provided by comparing with the outcome from the deterministic model. The detailed result file from OPL is attached in Appendix C.

6.1 Stochastic Model Solution

The solution provided by the stochastic model is presented in Table 6.1.1. As shown in the scenario tree (See Figure 5.1.2), the node representing the first-stage decision diverges into three branches. Those three branches diverge again into another three branches at the second-stage decisions. At the end, there are in total nine branches, meaning that there are nine possible scenarios with equal probabilities.

The following sections examine the decision variables at objective function, first-stage decision, amount of surplus at each period, and the second-stage decision. The results of the every branch will be examined. Table 6.1.2 includes the details of the surplus at every branch for all the periods

6.1.1 Objective Value

The objective value of the stochastic model is $7648.8. The initial wealth is set reasonably (W = $90,000) given that the sum of the total liability is $104,000. As expected, there are no shortages in the result; both $Q_{ts}$ and $QQQQ_{iks}$ are zero. Thus, the objective value purely represents the
surplus at the end of $t = 6$. The total cost to meet the liability, therefore, is, $90,000 - 7648.8 = $82351.2. If any investment is used to meet the liability, exactly $104,000 would have to be spent. Thus, the bond investment through the stochastic model saved, $104,000 - $82351.2 = $21648.8.

6.1.2 First-Stage Decision
At the first decision stage, the bond index and the amount to be purchased are determined. This decision is determined under the uncertainty of the bond trading rate. At this stage, three stocks were chosen: bond 1 ($X_1 = 177$), bond 11 ($X_{11} = 363$), and bond 12 ($X_{12} = 151$). The fact that there are no bonds selected from range III clearly explains that the stochastic model takes a hedging effect.

Among the bonds in range I, bond 1 was selected due to its short maturity, $M_1 = 3$, along with high coupon rate. After purchasing the bonds with pre-defined initial wealth, $Z_0 = 22376.4$ is left. To guarantee that the liabilities are paid for $t = 1, 2, 3$ in any scenario, the stochastic model takes a hedge effect by choosing the bond that provides the principal back at $t = 3$ and has a default rate of zero.

The bond 11 has the highest default rate ($D_{11} = 0.08$) among range II bonds, but it also has the highest coupon rate ($C_{11} = 0.08$). When comparing with the default rate of the bonds in range III, however, the risk is ignorable. Also, bond 12 offsets the risks since bond 12 has a lower coupon rate with a low default rate.
6.1.3 Portfolio Performance during First-Periods (t = 1, 2, 3)

After the first-stage decision, the liabilities for the first three periods are paid. Under the assumption (Section 5.1), the bond trading rate can go higher, stay the same, or go lower with the equal probabilities. It should be noted that the bond trading rate itself does not directly affect the performance during this period, because no bonds are traded. Rather it implies the changes in default penalty. All scenario branches start with $Z_0 = 22376.4$.

6.1.3.1 Portfolio Performance during First-Periods: T = 1.3

When the upper branch of the scenario tree is taken, that is when $T = 1.3$, the default penalty is $(1-D)$. This is the worst default penalty. Thus, the least amount of surplus is left compared to other branches. After $t = 1$, $13188.2$ is left (i.e. $Z_{11} = 13188.2$). For $t = 2, 3$, no surplus is left (i.e. $Z_{21} = Z_{31} = 0$); all the cash inflows, including the coupon payment and the principal payment, are used to meet the liabilities. The principal payment of bond 1 at $t = 3$ has prevented the model from having shortage.

6.1.3.2 Portfolio Performance during First-Periods: T = 1.0

When the middle branch ($T = 1.0$) is taken, the default penalty is $(1 - D^3)$. Thus, the coupon payment and the principal payment are relatively better than the first scenario. The surplus at each period are: $Z_{12} = 13425.6$, $Z_{22} = 474.8$, $Z_{32} = 712.2$. Again like in the upper branch, $Z_{12}$ and $Z_{22}$ keep decreasing, implying that the coupon payments are not enough to meet the liability and thus surplus from the previous periods are used to meet the liabilities. However, the
surplus at $t = 3$ increases due to the principal payment of bond 1. Part of the principal is used to pay off the liability, and the rest is accumulated as a surplus.

### 6.1.3.3 Portfolio Performance during First-Periods: $T = 0.7$

The lower branch ($T = 0.7$) has the lowest default penalty, which is $(1 - D^3)$. Thus, the amount of the surplus is the most among the three possible branches, demonstrating that the portfolio has performed best in this branch up until $t = 3$. The surplus at each period are: $ZZ_{13} = $13443.3, $ZZ_{23} = $510.2, $ZZ_{33} = $765.3. The principal payment of bond 1 at $t = 3$ helps to accumulate the surplus in the same manner as it does in the upper branches.

### 6.1.4 Second-Stage Decision

By the time when the second-stage decision is going to be made, the bond trading rate of the previous periods ($t = 1, 2, 3$) are known. Thus, the portfolio is rebalanced to react to the outcomes of the first -periods.

### 6.1.4.1 Second-Stage Decision: $T = 1.3$

The performance of the upper branch ($T = 1.3$) has performed the worst until $t = 3$. It has zero surpluses before the rebalancing. After the rebalancing, the portfolio has $66827.3. To adjust the portfolio, all of bond 11 and bond 12 are sold. There are two reasons for this; 1) the upper branch has received the most severe penalty by receiving only $(1 - D)$ of coupon payment or principal payment, 2) the bond trading rate is high in this branch, thus premium is added up when the bonds are sold. The outcome surplus proves that the stochastic model has adjusted accordingly.
6.1.4.2 Second-Stage Decision: T = 1.0
The adjustment of the middle branch (T = 1.0) has some commonality as well as difference with the upper branch adjustment. All of bonds 11 are sold, but all of bonds 12 are kept, and additional 261 units of bond 1 are purchased. The rationality for this reaction is as follows. Not as severe as the upper branch, the middle branch has been affected to the default penalty by receiving only \((1 - D^2)\) % of the payment. Thus the portfolio reacts to sell bond 11 which has the higher default rate of all three purchased at first-stage decision. Since no premium is paid, the portfolio has compensated the shortages by purchasing additional bond 1. The bond 1 has chosen because of the following factors: no default penalty, three years of maturity, and relatively low face value. As a result, the surplus is increased from $712.2 to $13393.2 after rebalance.

6.1.4.3 Second-Stage Decision: T = 0.7
The lower branch (T = 0.7) behaves in a slightly different way. Majority of bond 11 are sold, and all of bond 12 are kept. Also, additional 251 units of bond 1 and 156 units of bond 14 are purchased. It is interesting to see that even the past history provided the least default penalty, which is \((1 - D^3)\), the portfolio does not include any of range III bonds. Instead, the portfolio purchases additional bond 1 which provides lower coupon rate. The lower default rate however takes a risk of choosing bond 14 which has the worst default rate but the highest coupon rate among the bonds in range II. At the end of the rebalancing, no surplus is left. It is quite contrasting to the case in the upper branch. The upper branch has zero surpluses before the rebalancing but ended up with surplus of $66827.3, which is the greatest of all three. The lower branch however previously has the surplus of $765.3, which is the greatest of all three, but ended up with having zero surpluses.
6.1.5 Portfolio Performance during the Second-Periods (t = 4, 5, 6)

After the rebalance at the second-stage decision, each node diverges into three branches again. Thus, three distinctive surpluses at the rebalancing node would diverge into nine different surpluses. The following sections group the scenarios into three: scenario 1-3, scenario 4-6, and scenario 7-9 (See Figure 6.1.5).

6.1.5.1 Portfolio Performance during the Second-Periods: scenario 1-3

The rebalanced portfolio before t = 4 for this group has no bonds; the coupon payment periods for bond 1 has expired by t = 3, thus the principal is paid, and all of bond 11 and bond 12 are sold. The surplus of $66827.3 is the only thing that the portfolio carries. As a result, there are no cash inflows for t = 4, 5, 6. However, the cash flow matching constraints are still satisfied since the portfolio has enough surpluses to meet the cash outflow. The fact that no bonds are invested at t = 4, 5, 6 makes the trading rate negligible, thus sets the surpluses identical for scenarios 1 – 3. The results are as follows: $ZZZZ_{4k1} = $50827.3 ($66827.3 - $16000), $ZZZZ_{5k1} = $34827.3 ($50827.3 - $16000), $ZZZZ_{6k1} = $14827.3 ($34827.3 - $20000), where $k = 1, 2, 3$.

6.1.5.2 Portfolio Performance during the Second-Periods: scenario 4-6

The rebalanced portfolio for scenario 4 - 6 has purchased additional 261 units of bond 1, sold all of bond 11, and kept all of bond 12. The rebalanced surplus is $13393.2.

When the upper branch (T = 1.3) is taken in the second-periods, the default penalty of (1 – D) % is taken. Thus, bond 12 which has a penalty rate of 0.02 affects from this default penalty. The surplus for t = 4 is zero, indicating that the cash inflow from the coupon payment is not enough
to meet the liability; thus, the surplus from the previous periods are used. However, at $t = 5$, the principal payment of bond 12 and the coupon payment of bond 1 provide enough capital to meet the liability. At $t = 6$, the principal payment of bond contributes to pay the liability and makes the surplus of $3949.9$

The similar cash inflows and cash outflows occur for both the middle branch ($T = 1.0$) and the lower branch ($T = 0.7$), except that the different default penalty is in effect. Since the default penalty alleviates as the trading rate decreases, the portfolio performs better compared to the upper branch; the lower branch performs the best.

6.1.5.3 Portfolio Performance during the Second-Periods: scenario 7 – 9

The rebalanced portfolio for scenarios 7-9 adjust in the following way during the rebalancing; it purchases additional bond 1 and bond 14, sell most of bond 11 (150 units out of 151 sold), and keep all of bond 12, leaving surplus of $0$.

Scenario 7 happens when the first branch is taken. The bond trading rate is $T = 1.3$, implying that the default penalty is $(1 - D) \%$. Even though $ZZZ_{33} = 0$, the cash flow matching constraints are still satisfied due to mostly the principal payment of bond 14 and the coupon payment of bond 12. All the cash inflows at $t = 4$ are used to meet the liability, leaving zero surpluses. However, the cash flow constraint at $t = 5$ is satisfied again due to the principal payment of bond 12 this time. All of the cash inflows are used again. But, scenario 7 ends up having the surplus of $3046.6$, because bond 1 returns the principal back at $t = 6$. 
Again, the similar cash inflows/outflows occur for the scenario 8 and 9. The only difference is the penalty on the default rate. Thus, scenario 8 and 9 perform slightly better than scenario 7, scenario 9 performing the best.

6.2 Calculating VSS

To validate the algorithm, VSS is calculated (See Section 4.5). The expected trading rate can be calculated as follows:

\[ P_1T_1 + P_2T_2 + P_3T_3 = 1.0 \]

The deterministic solution using the expected trading rate suggests purchasing 172 units of bond 1 and 520 units of bond 11 at the first-stage decision. (See Appendix D) This solution however does not provide enough cash inflow or surplus to meet the liabilities for the first periods, depending on the trading path it takes. For example, when \( T = 1.3 \), shortage of $619 occurs at \( t = 1 \). This shortage generates the loss in the objective value. Additionally, less cash inflows caused when \( T = 1.3 \) diminishes \( ZZZZ_{6x} \), decreasing the objective value. When this solution is asserted into the stochastic model as the first-stage decision, $6818.2 turns out to be the objective value. Thus,

\[ VSS = $7648.8 - $6818.2 = $830.6 \]

This value, $830.6, represents the value of knowing the possible randomness and its probability distribution, or loss by not considering possible scenarios. The loss consists of the shortage
penalty as well as default penalty. The detailed result obtained from OPL is presented in Appendix E.

6.3 Calculating EVPI

It is obvious that having perfect information would help in decision making by providing the right approach. However, there are few cases where perfect information is available before the decision making.

The calculation of EVPI requires the averages of the deterministic model. Since there are nine possible scenarios, nine different models are created. The details of the deterministic model are covered in the Section 6.4. The average of nine objective values of the deterministic model is $13721.4. Thus,

\[
EVPI = \$13721.4 - \$7648.8 = \$6072.6
\]

Thus it would be worth to pay at most $6072.6 to get the perfect information about the trading rate if the perfect information is available.

6.4 General Description of the Deterministic Model

In the model, both the first-stage decision and the second-stage decision are made under the uncertainty of bond trading rate for the upcoming periods. The deterministic model assumes that the trading rate will take a certain path all the time. As long as the assumption is valid, the
model should work fine, providing more advantages over the stochastic model. The problem is
that this assumption is not accurate in bond trading market.

Unlike the stochastic model, deterministic model does not provide any hedge effect. The
following examples demonstrate this point. It should be noted that only very distinctive
examples are listed here. Appendix F presents the details of the deterministic model.

- When it is determined that scenario 1 will occur, the portfolio selects mix of bonds in
  range I and II, but sells all range II bonds during the rebalancing.
- When it is determined that scenario 9 will occur, the portfolio selects only risky bonds
  (range III). During the rebalancing, the portfolio purchases even additional units of
  bonds in range III.

When the above solutions are used in the stochastic model, it generates loss in the profit caused
by unexpected events that are not captured in the deterministic model.
7 Model Evaluation

This chapter evaluates the final model and diagnose if the model satisfies the primary/secondary goals and objectives of the project.

7.1 Primary Goal Evaluation

The primary goal of this project is to develop the stochastic bond portfolio optimization model. Chapter6 proves that the stochastic model has more advantages over the deterministic model given that the uncertainty exists at the time of decision-making. The following points demonstrate how the model has (or not) met the objectives related to the primary goal. The number in the bracket indicates the objective number in Chapter 3.

- (3.1.1) Portfolio generated by the stochastic model does not include any bonds in range III in both first-stage decision and the second-stage decision.(See Chapter 6)
- (3.1.2) VSS = $830.6 (See Section 6.2)
- (3.1.3) As long as there is no shortage, meaning that both QQ_{ts} and QQQQ_{s,t} are zero, the initial wealth will guarantee that the liability is met. It is recommended to use an initial wealth little less than the sum of the liability, and then reduce the initial wealth just before any shortage occurs.
7.2 Secondary Goal Evaluation

The secondary goal of this project is to develop the model that resembles real life as much as possible. This goal is met, since the stochastic model correctly depicts the relationship between the bond trading rate, which is affected by interest rate, and the default penalty. The objectives related to the secondary goal are, however, partially met.

- (3.2.1) It is assumed that there are only nine scenarios possible. However, real bond market is composed of multifarious scenarios.
- (3.2.2) Original data is adopted from [4], but part of the data is modified because not all the bonds have the linear relationship between the default rate and the coupon rate.
8 Future Considerations

The current stochastic model behaves in the expected way, satisfying both the primary and secondary goals. However, it is hard to say that the current model closely resembles the real life for the following reasons:

- It is assumed that the bond trading rates during the first three years \((t = 1, 2, 3)\) and the second three years \((t = 4, 5, 6)\) do not change. However, it is quite possible that the bond trading rate deviates in different ways even during one year.
- The assumption that the scenario tree diverges only into three branches does not sufficiently count the variability of the bond trading rate.
- The assumption that the distribution of the bond trading rate is equal does not illustrate real life correctly.
- Many uncertainties other than bond trading rate exist in bond market.
- The default penalty is not set methodically.
- The original data is manipulated.

The following points describe the necessary steps to improve the current model.

- Volatility of the bond trading rate needs to be implemented by allowing that the bond trading rate can deviate even in a single period.
- Additional randomness other than bond trading rate needs to be implemented.
• Variability of randomness (i.e. bond trading rate) and correct distribution need to be implemented.

• More research on the relationship between default rates and interest rates are necessary to apply the default penalty correctly
9 Conclusion

The goal of this project was to develop the 2-stage stochastic model that optimizes bond portfolio. The constructed portfolio suggests how to invest the pre-defined initial wealth at the first-stage decision and how to rebalance the portfolio at the second-stage decision to meet liabilities. The volatility as well as variability of the bond trading rate, which is uncertain at the decision points, necessitates the use of the stochastic programming model.

The initial model was written in mathematical format then converted to OPL. Through model verification and validation process, the model continued to be revised and the data was manipulated (See Chapter 4).

The stochastic optimization model provides solution that takes hedge effects by selecting bonds in either range I or II. This solution contrasts to the deterministic solution which does not consider the variety of randomness. The benefit of considering randomness is clearly displayed in VSS (See Chapter 6).

Even though the model constructs a solution that maximizes the surplus at the end of the timeline, there are areas of improvement in this model (See Chapter 8). Rigorous research in randomness in bond market and its probability distribution would stimulate preciseness of the model, validating the making the current model enhance the current model.
10 References


11 Figures

Figure 4.6.1.1 Coupon Rate vs. Default Rate

![Coupon Rate vs. Default Rate](image-url)
Figure 4.6.1.2 Face Value vs. Default Percentage (%)
Figure 5.1.1 Timeline of the Model

L1  L2  L3  L4  L5  L6

T = 0  1  2  3  4  5  6

1\textsuperscript{st} Stage  2\textsuperscript{nd} Stage
Figure 5.1.2 Scenario Tree with the timeline
Figure 5.4 Concave Utility Function of Surplus and Shortage

Utility Function of Surplus and Shortage

Utility

Wealth

Goal

4

1
Figure 6.1.5 Scenario Tree with Scenario Numbers
Table 4.6.1 Finalized Bond Data

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<td>92.25</td>
<td>85.58</td>
<td>71.23</td>
<td>72.411</td>
</tr>
<tr>
<td>C</td>
<td>0.0885</td>
<td>0.1</td>
<td>0.1025</td>
<td>0.09</td>
<td>0.07</td>
<td>0.09</td>
<td>0.0825</td>
<td>0.1015</td>
<td>0.0922</td>
<td>0.0925</td>
</tr>
<tr>
<td>D</td>
<td>0.35</td>
<td>0.38</td>
<td>0.51</td>
<td>0.45</td>
<td>0.25</td>
<td>0.48</td>
<td>0.4</td>
<td>0.65</td>
<td>0.62</td>
<td>0.5</td>
</tr>
<tr>
<td>M</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>26</td>
<td>27</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>
Table 5.3.1 Decision Variables at Each Period

<table>
<thead>
<tr>
<th>1st - stage</th>
<th>T = 1</th>
<th>T = 2</th>
<th>T = 3</th>
<th>2nd - stage</th>
<th>T = 4</th>
<th>T = 5</th>
<th>T = 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z_0</td>
<td>ZZ_{1s}</td>
<td>ZZ_{2s}</td>
<td>ZZ_{3s}</td>
<td>ZZZ_{3s}</td>
<td>ZZZZ_{4ks}</td>
<td>ZZZZ_{5ks}</td>
<td>ZZZZ_{6ks}</td>
</tr>
<tr>
<td>X_i</td>
<td></td>
<td></td>
<td></td>
<td>Y_{is}, K_{is}, S_{is}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Qi</td>
<td>QQ_{1s}</td>
<td>QQ_{2s}</td>
<td>QQ_{3s}</td>
<td></td>
<td>QQQQ_{4ks}</td>
<td>QQQQ_{5ks}</td>
<td>QQQQ_{6ks}</td>
</tr>
</tbody>
</table>
Table 6.1.1 Stochastic Solution

<table>
<thead>
<tr>
<th>Decision Stage, Scenario</th>
<th>Range I</th>
<th>Range II</th>
<th>Range III</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 1-9</td>
<td>$X_1 = 176.7$</td>
<td>$X_{11} = 362.5$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$X_{12} = 150.7$</td>
<td></td>
</tr>
<tr>
<td>2, 1-3</td>
<td></td>
<td>$S_{11} = 362.5$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$S_{12} = 150.7$</td>
<td></td>
</tr>
<tr>
<td>2, 4-6</td>
<td>$Y_1 = 261$</td>
<td>$S_{11} = 362.5$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$K_{12} = 150.7$</td>
<td></td>
</tr>
<tr>
<td>2, 7-9</td>
<td>$Y_1 = 251.2$</td>
<td></td>
<td>$Y_{14} = 155.6$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$K_{11} = 0.62$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$K_{12} = 150.7$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$S_{11} = 361.9$</td>
</tr>
</tbody>
</table>
Table 6.1.2 Surplus at Each Period (unit: $)

<table>
<thead>
<tr>
<th></th>
<th>1st stage</th>
<th>t = 1</th>
<th>t = 2</th>
<th>t = 3</th>
<th>2nd stage</th>
<th>t = 4</th>
<th>t = 5</th>
<th>t = 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>sc1</td>
<td>22376.4</td>
<td>13188.2</td>
<td>0</td>
<td>0</td>
<td>66827.3</td>
<td>50827.3</td>
<td>34827.3</td>
<td>14827.3</td>
</tr>
<tr>
<td>sc2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>50827.3</td>
<td>34827.3</td>
<td>14827.3</td>
</tr>
<tr>
<td>sc3</td>
<td>13425.6</td>
<td>474.8</td>
<td>712.2</td>
<td>13393.2</td>
<td></td>
<td>21.7</td>
<td>311.3</td>
<td>4261.2</td>
</tr>
<tr>
<td>sc4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>22.2</td>
<td>317.5</td>
<td>4267.4</td>
</tr>
<tr>
<td>sc5</td>
<td>13443.3</td>
<td>510.2</td>
<td>765.3</td>
<td>0</td>
<td></td>
<td>1097.8</td>
<td>1391.9</td>
<td>4438.5</td>
</tr>
<tr>
<td>sc6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1184.3</td>
<td>1484.6</td>
<td>4531.2</td>
</tr>
</tbody>
</table>
13 Appendices

Appendix A – OPL Code on Data File

```
new_period = 6;
nbScenario = 3;
W = 90000;
bond_index = 30;

//liability
L = [0, 14000.0, 18000.0, 20000.0, 16000.0, 16000.0, 20000.0];

//bond trading rate
T = [1.3, 1.0, 0.7];

//probability of scenario happen
P = [0.333, 0.333, 0.333];

//default rate of bond i
D = [0, 0.08, 0.02, 0.015, 0.08, 0.05, 0.05, 0.012, 0.01, 0.05, 0.01, 0.35, 0.38, 0.51, 0.45, 0.25, 0.48, 0.4, 0.65, 0.62, 0.5];

//maturity year of bond i
M = [3.00, 28.00, 5.00, 5.00, 4.00, 8.0, 27.00, 27.00, 5.0, 5.00, 5.00, 21.00, 1.00, 4.00, 6.00, 4.00, 5.00, 5.00, 7.0, 2.00, 3.0, 6.0, 4.00, 5.00, 5.0, 26.00, 27.00, 3.00, 4.00];

//face value of bond i
F = [91.750, 93.153, 90.625, 93.026, 100.625, 94.548, 94.360, 97.763, 98.67, 92.499, 101.05, 98.025, 100.063, 93.938, 79.375, 123.56, 91.250, 100.000, 100.0, 106.281, 85.616, 84.024, 97.748, 96.000, 98.905, 74.226, 92.250, 85.580, 71.23, 72.411];

//coupon rate of bond i
C = [0.0635, 0.064, 0.0622, 0.0625, 0.072, 0.06, 0.065, 0.0645, 0.055, 0.0577, 0.08, 0.075, 0.0745, 0.0795, 0.0735, 0.07175, 0.0725, 0.0725, 0.0675, 0.06625, 0.0885, 0.10, 0.1025, 0.09, 0.07, 0.09, 0.0825, 0.1015, 0.0922, 0.0925];
```
Appendix B – OPL Code on Mod File

//scenario - for the both first-stage decision and second-stage decision, there are three possible
//bond trading rate branch
int nbScenario = ...;
range scenario 1..nbScenario;

// bond index
int bond_index = ...;
range bond 1..bond_index;

//period
int new_period = ...;
range period 0..new_period;

float L[period]=...;       //liability of the period t
float T[scenario] =...;    //bond trading rate at scenario s
float P[scenario] = ...;   //probability of scenario s happen
float F[bond]=...;         //face value of bond i
float D[bond] =...;        //probability of default of bond i
float M[bond] =...;        //maturity year of bond i
float C[bond]=...;         //coupon rate of bond i
float W =...;              //initial wealth at stage 0

var float+ X[bond];                  //amount of bond i purchased at stage 0
var float+ Y[bond, scenario];        //amount of bond i purchased at stage 1
var float+ S[bond, scenario];        //amount of bond i sold at stage 1
var float+ K[bond, scenario];        //amount of bond i kept at stage 1
var float+ Z[period];                //amount of surplus at $t = 0$
var float+ ZZ[period, scenario];     //amount of surplus at $t = 1,2,3$
var float+ ZZZ[ period, scenario];   //amount of surplus at the end of period 3 after rebalancing
var float+ ZZZZ[period, scenario, scenario]; //amount of surplus at $t = 4,5,6$
var float+ QQ[period, scenario];     //amount of shortage at $t = 1,2,3$
var float+ QQQQ[period, scenario, scenario];        //amount of shortage at $t = 4,5,6$
maximize
sum(s in scenario, k in scenario)(ZZZZ[6,k,s]*P[s]*P[k]) - sum(t in period, s in scenario, k in scenario)(4*QQQQ[t,k,s]*P[s]*P[k])
-sum(t in period, s in scenario)(4*QQ[t,s]*P[s])

subject to {
// the purchased bond at t=0 should be within the initial wealth limit
sum(i in bond)F[i]*X[i] + Z[0] = W;
Z[0] = ZZ[0,1] = ZZ[0,2] = ZZ[0,3];

// cash flow constraint for t = 1, 2, 3
// T = 1.3
forall (t in period, s in scenario)
if (1 <= t <= 3 & s = 1) then
   sum (i in bond : M[i]>t)(1-D[i])*C[i]*F[i]*X[i] + sum(i in bond : M[i] = t)F[i]*(1-D[i])*X[i] + ZZ[t-1,s] - ZZ[t,s] + QQ[t,s] = L[t]
endif;
// T = 1.0
forall (t in period, s in scenario)
if (1 <= t <= 3 & s = 2) then
   sum (i in bond : M[i]>t)(1-D[i]*D[i])*C[i]*F[i]*X[i] + sum(i in bond : M[i] = t)F[i]*(1-D[i]*D[i])*X[i] + ZZ[t-1,s] - ZZ[t,s] + QQ[t,s] = L[t]
endif;
// T = 0.7
forall (t in period, s in scenario)
if (1 <= t <= 3 & s = 3) then
   sum (i in bond : M[i]>t)(1-D[i]*D[i]*D[i])*C[i]*F[i]*X[i] + sum(i in bond : M[i] = t)F[i]*(1-D[i]*D[i]*D[i])*X[i] + ZZ[t-1,s] - ZZ[t,s] + QQ[t,s] = L[t]
endif;
forall (i in bond, s in scenario : M[i]>3)
X[i] - S[i,s] = K[i,s];

// rebalancing
forall (s in scenario)
ZZ[3,s] + sum(i in bond : M[i] >3)F[i]*S[i,s]*T[s] - sum(i in bond)F[i]*Y[i,s]*T[s] = ZZZ[3,s];
// s being the previous scenario, k being the current scenario
forall (s in scenario)
ZZZ[3,s] = ZZZZ[3, 1,s] = ZZZZ[3,2,s] = ZZZZ[3,3,s];

// cash flow constraint for t = 4, 5, 6
// T = 1.3
forall (t in period, s in scenario, k in scenario)
if (4 <= t <= 6 & k = 1) then
  sum (i in bond : M[i] > t)C[i]*(1-D[i])*F[i]*K[i,s]
  + sum (i in bond : M[i] > t-3)C[i]*(1-D[i])*F[i]*Y[i,s]
  + sum (i in bond : M[i] = t)C[i]*(1-D[i])*F[i]*K[i,s]
  + sum (i in bond : M[i] = t-3)C[i]*(1-D[i])*F[i]*Y[i,s] + ZZZZ[t-1,k,s] - ZZZZ[t,k,s]+
  QQQ[t,k,s]= L[t]
endif;

// T = 1.0
forall (t in period, s in scenario, k in scenario)
if (4 <= t <= 6 & k = 2) then
  sum (i in bond : M[i] > t)C[i]*(1-D[i]*D[i])*F[i]*K[i,s]
  + sum (i in bond : M[i] > t-3)C[i]*(1-D[i]*D[i])*F[i]*Y[i,s]
  + sum (i in bond : M[i] = t)C[i]*(1-D[i]*D[i])*F[i]*K[i,s]
  + sum (i in bond : M[i] = t-3)C[i]*(1-D[i]*D[i])*F[i]*Y[i,s] + ZZZZ[t-1,k,s] - ZZZZ[t,k,s]+
  QQQ[t,k,s]= L[t]
endif;

// T = 0.7
forall (t in period, s in scenario, k in scenario)
if (4 <= t <= 6 & k = 3) then
  sum (i in bond : M[i] > t)C[i]*(1-D[i]*D[i]*D[i])*F[i]*K[i,s]
  + sum (i in bond : M[i] > t-3)C[i]*(1-D[i]*D[i]*D[i])*F[i]*Y[i,s]
  + sum (i in bond : M[i] = t)C[i]*(1-D[i]*D[i]*D[i])*F[i]*K[i,s]
  + sum (i in bond : M[i] = t-3)C[i]*(1-D[i]*D[i]*D[i])*F[i]*Y[i,s] + ZZZZ[t-1,k,s] - ZZZZ[t,k,s]+
  QQQ[t,k,s]= L[t]
endif;

};
Appendix C – Stochastic Model OPL Result

Decision variables with a value of 0 are omitted.

Optimal Solution with Objective Value: 7648.7726

\[ X[1] = 176.7632 \]
\[ X[12] = 150.7235 \]

\[ Y[1,2] = 261.0344 \]
\[ Y[1,3] = 251.1892 \]
\[ Y[14,3] = 155.5838 \]

\[ S[11,1] = 362.5031 \]
\[ S[11,2] = 362.5031 \]
\[ S[11,3] = 361.8861 \]
\[ S[12,1] = 150.7235 \]
\[ K[11,3] = 0.6169 \]
\[ K[12,2] = 150.7235 \]
\[ K[12,3] = 150.7235 \]

\[ Z[0] = 22376.3591 \]

\[ ZZ[1,1] = 13188.1795 \]
\[ ZZ[1,2] = 13425.5813 \]
\[ ZZ[1,3] = 13443.2703 \]
\[ ZZ[2,1] = 0.0000 \]
\[ ZZ[2,2] = 474.8035 \]
\[ ZZ[2,3] = 510.1815 \]
\[ ZZ[3,1] = 0.0000 \]
\[ ZZ[3,2] = 712.2053 \]
\[ ZZ[3,3] = 765.2723 \]

\[ ZZZ[3,1] = 66827.3019 \]
\[ ZZZ[3,2] = 13393.2424 \]
\[ ZZZ[3,3] = 0.0000 \]

\[ ZZZZ[4,1,1] = 50827.3019 \]
\[ ZZZZ[4,1,2] = 0.0000 \]
\[ ZZZZ[4,1,3] = 0.0000 \]
\[ ZZZZ[4,2,1] = 50827.3019 \]
\[ ZZZZ[4,2,2] = 21.7188 \]
\\[ZZZZ[4,2,3] = 1097.7669\\]
\\[ZZZZ[4,3,1] = 50827.3019\\]
\\[ZZZZ[4,3,2] = 22.1531\\]
\\[ZZZZ[4,3,3] = 1184.2851\\]
\\[ZZZZ[5,1,1] = 34827.3019\\]
\\[ZZZZ[5,1,2] = 0.0000\\]
\\[ZZZZ[5,1,3] = 0.0000\\]
\\[ZZZZ[5,2,1] = 34827.3019\\]
\\[ZZZZ[5,2,2] = 311.3024\\]
\\[ZZZZ[5,2,3] = 1391.9393\\]
\\[ZZZZ[5,3,1] = 34827.3019\\]
\\[ZZZZ[5,3,2] = 317.5284\\]
\\[ZZZZ[5,3,3] = 1484.6163\\]
\\[ZZZZ[6,1,1] = 14827.3019\\]
\\[ZZZZ[6,1,2] = 3949.9052\\]
\\[ZZZZ[6,1,3] = 3046.6101\\]
\\[ZZZZ[6,2,1] = 14827.3019\\]
\\[ZZZZ[6,2,2] = 4261.2076\\]
\\[ZZZZ[6,2,3] = 4438.5493\\]
\\[ZZZZ[6,3,1] = 14827.3019\\]
\\[ZZZZ[6,3,2] = 4267.4336\\]
\\[ZZZZ[6,3,3] = 4531.2263\\]
Appendix D– Deterministic Model Solution when Expected Bond Trading Rate is used

Decision variables with a value of 0 are omitted.

Optimal Solution with Objective Value: 4741.6264

\[
\begin{align*}
X[1] &= 172.4671 \\
Y[1] &= 269.6635 \\
Z[0] &= 21638.0788 \\
ZZ[1] &= 12819.0394 \\
ZZ[2] &= 0.0000 \\
ZZ[3] &= 0.0000 \\
ZZZ[3] &= 13274.5942 \\
ZZZZ[4] &= 0.0000 \\
ZZZZ[5] &= 0.0000 \\
ZZZZ[6] &= 4741.6264
\end{align*}
\]
Appendix E – Stochastic Model Solution using Expected Value Solution

Decision variables with a value of 0 are omitted.

Optimal Solution with Objective Value: 6878.1746

\[
\begin{align*}
Y[1,2] &= 255.1521 \\
Y[1,3] &= 241.8966 \\
S[11,1] &= 519.9215 \\
S[11,2] &= 363.8058 \\
S[11,3] &= 362.9751 \\
K[11,2] &= 156.1157 \\
K[11,3] &= 156.9464 \\
Z[0] &= 21638.0760 \\
ZZ[1,1] &= 13128.3833 \\
ZZ[1,2] &= 12819.0392 \\
ZZ[1,3] &= 12843.7843 \\
ZZ[2,1] &= 0.0000 \\
ZZ[2,2] &= 0.0000 \\
ZZ[2,3] &= 49.4927 \\
ZZ[3,1] &= 0.0000 \\
ZZ[3,2] &= 0.0023 \\
ZZ[3,3] &= 74.2425 \\
ZZZ[3,1] &= 68299.4878 \\
ZZZ[3,2] &= 13352.3760 \\
ZZZ[3,3] &= 0.0000 \\
ZZZZ[4,1,1] &= 52299.4878 \\
ZZZZ[4,1,2] &= 0.0000 \\
ZZZZ[4,1,3] &= 0.0000 \\
ZZZZ[4,2,1] &= 52299.4878 \\
ZZZZ[4,2,2] &= 92.8861 \\
ZZZZ[4,2,3] &= 1167.2544 \\
ZZZZ[4,3,1] &= 52299.4878 \\
ZZZZ[4,3,2] &= 100.3170 \\
ZZZZ[4,3,3] &= 1260.6348 \\
ZZZZ[5,1,1] &= 36299.4878 \\
ZZZZ[5,1,2] &= 0.0000 \\
ZZZZ[5,1,3] &= 0.0000 \\
ZZZZ[5,2,1] &= 36299.4878 \\
ZZZZ[5,2,2] &= 1253.9623 \\
ZZZZ[5,2,3] &= 2334.5088 
\end{align*}
\]
$ZZZZ[5,3,1] = 36299.4878$
$ZZZZ[5,3,2] = 1354.2792$
$ZZZZ[5,3,3] = 2521.2695$
$ZZZZ[6,1,1] = 16299.4878$
$ZZZZ[6,1,2] = 3410.2025$
$ZZZZ[6,1,3] = 2194.0132$
$ZZZZ[6,2,1] = 16299.4878$
$ZZZZ[6,2,2] = 4664.1648$
$ZZZZ[6,2,3] = 4528.5220$
$ZZZZ[6,3,1] = 16299.4878$

$QQ[1,1] = 618.6907$
$QQ[1,2] = 0.0024$
$QQ[3,1] = 309.3418$
Appendix F – Deterministic Model Solution

Decision variables with a value of 0 are omitted.

The deterministic solutions are in order of the scenario number. For example, if the deterministic model is made under the assumption that the trading rate will be 1.3 for the entire periods, it will be designated as scenario 1.

1. Scenario 1

Optimal Solution with Objective Value: 14834.2998

\[ X[1] = 176.7428 \]
\[ X[11] = 508.7679 \]
\[ S[11] = 508.7679 \]
\[ Z[0] = 22372.8497 \]
\[ ZZ[1] = 13186.4249 \]
\[ ZZ[2] = 0.0000 \]
\[ ZZ[3] = 0.0000 \]
\[ ZZZ[3] = 66834.2998 \]
\[ ZZZZ[4] = 50834.2998 \]
\[ ZZZZ[5] = 34834.2998 \]
\[ ZZZZ[6] = 14834.2998 \]

2. Scenario 2

Optimal Solution with Objective Value: 14834.2998

\[ X[1] = 176.7428 \]
\[ X[11] = 508.7679 \]
\[ S[11] = 508.7679 \]
\[ Z[0] = 22372.8497 \]
\[ ZZ[1] = 13186.4249 \]
\[ ZZ[2] = 0.0000 \]
\[ ZZ[3] = 0.0000 \]
\[ ZZZ[3] = 66834.2998 \]
\[ ZZZZ[4] = 50834.2998 \]
\[ ZZZZ[5] = 34834.2998 \]
\[ ZZZZ[6] = 14834.2998 \]
3. Scenario 3

Optimal Solution with Objective Value: 14834.2998

\[
\begin{align*}
X[1] &= 176.7428 \\
Z[0] &= 22372.8497 \\
ZZ[1] &= 13186.4249 \\
ZZ[2] &= 0.0000 \\
ZZ[3] &= 0.0000 \\
ZZZ[3] &= 66834.2998 \\
ZZZZ[4] &= 50834.2998 \\
ZZZZ[5] &= 34834.2998 \\
ZZZZ[6] &= 14834.2998
\end{align*}
\]

4. Scenario 4

Optimal Solution with Objective Value: 4146.0590

\[
\begin{align*}
X[1] &= 173.4091 \\
X[12] &= 150.5939 \\
Y[1] &= 263.1723 \\
K[12] &= 150.5939 \\
Z[0] &= 21799.9682 \\
ZZ[1] &= 12899.9841 \\
ZZ[2] &= 0.0000 \\
ZZ[3] &= 0.0000 \\
ZZZ[3] &= 13381.7209 \\
ZZZZ[4] &= 0.0000 \\
ZZZZ[5] &= 0.0000 \\
ZZZZ[6] &= 4146.0590
\end{align*}
\]
5. Scenario 5

Optimal Solution with Objective Value: 4741.6264

\[X[1] = 172.4671\]
\[X[11] = 519.9215\]

\[Y[1] = 269.6635\]
\[S[11] = 376.2120\]
\[K[11] = 143.7095\]

\[Z[0] = 21638.0788\]
\[ZZ[1] = 12819.0394\]
\[ZZ[2] = 0.0000\]
\[ZZ[3] = 0.0000\]
\[ZZZ[3] = 13274.5942\]
\[ZZZZ[4] = 0.0000\]
\[ZZZZ[5] = 0.0000\]
\[ZZZZ[6] = 4741.6264\]

6. Scenario 6

Optimal Solution with Objective Value: 5184.4189

\[X[1] = 172.4671\]
\[X[11] = 519.9215\]

\[Y[22] = 317.1305\]
\[K[11] = 133.4831\]

\[Z[0] = 21638.0788\]
\[ZZ[1] = 12819.0394\]
\[ZZ[2] = 0.0000\]
\[ZZ[3] = 0.0000\]
\[ZZZ[3] = 12403.0335\]
\[ZZZ[4] = 0.0000\]
\[ZZZ[5] = 0.0000\]
\[ZZ[6] = 4741.6264\]
\[ZZZZ[6] = 5184.4189\]
7. Scenario 7

Optimal Solution with Objective Value: 19472.1899

\[
\begin{align*}
X[21] &= 136.4764 \\
X[22] &= 858.3452 \\
Y[1] &= 430.2146 \\
Y[14] &= 312.2669 \\
Z[0] &= 6193.8383 \\
ZZ[1] &= 0.0000 \\
ZZ[2] &= 0.0000 \\
ZZ[3] &= 48164.1441 \\
ZZZ[3] &= 0.0000 \\
ZZZZ[4] &= 13493.5159 \\
ZZZZ[5] &= 0.0000 \\
ZZZZ[6] &= 19472.1899
\end{align*}
\]

8. Scenario 8

Optimal Solution with Objective Value: 21963.4921

\[
\begin{align*}
X[21] &= 136.4764 \\
X[22] &= 858.3452 \\
Y[1] &= 457.3678 \\
Y[14] &= 285.7462 \\
Z[0] &= 6193.8383 \\
ZZ[1] &= 0.0000 \\
ZZ[2] &= 0.0000 \\
ZZ[3] &= 48164.1441 \\
ZZZ[3] &= 0.0000 \\
ZZZZ[4] &= 13335.3182 \\
ZZZZ[5] &= 0.0000 \\
ZZZZ[6] &= 21963.4921
\end{align*}
\]
9. Scenario 9

Optimal Solution with Objective Value: 23481.5913

X[21] = 136.4764
X[22] = 858.3452

Y[14] = 113.1182
Y[21] = 142.1906
Y[22] = 547.5344

Z[0] = 6193.8383
ZZ[1] = 0.0000
ZZ[2] = 0.0000
ZZ[3] = 48164.1441
ZZZ[3] = 0.0000
ZZZZ[4] = 0.0000
ZZZZ[5] = 0.0000
ZZZZ[6] = 23481.5913