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CLASSIFICATION AND INTERPRETATION OF THE SINGULARITIES OF REDUNDANT MECHANISMS

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ABSTRACT

This paper introduces a unified approach to the singularity analysis of mechanisms with arbitrary kinematic chains and any number of inputs or outputs. The presented framework for the interpretation and classification of singularities is obtained by generalizing previous results (Zlatanov et al., 1995) valid only for non-redundant mechanisms. A velocity-space model is used to describe the instantaneous kinematics and interpret mechanism singularity. The definition of six singularity types and the study of their interdependence in the case of redundant mechanisms yield a comprehensive singularity classification.

1. INTRODUCTION

In a previous paper (Zlatanov et al., 1995), we presented a framework for the interpretation and classification of mechanism singularities. Singularity was defined in terms of the relationship of *all* of the joint velocities of the mechanism and the proposed classification of singularities (based on six newly-introduced singularity types) reflected the variety of kinematic implications of the degeneration of the instantaneous kinematics. This new approach to singularity analysis improves on previous methods based solely on input-output equations.

The results obtained by Zlatanov et al. (1995) are applicable to mechanisms with arbitrary kinematic chains with an equal number of inputs and outputs, i.e., to non-redundant mechanisms. The objective of the present paper is to further generalize the framework to include mechanisms with redundancy, leading to a truly comprehensive classification of all singularities of all mechanisms.

In Section 2, we introduce the necessary notation and terminology and briefly discuss the applicability of the velocity-equation and motion-space models of instantaneous kinematics for the case of redundant mechanisms. In Section 3, the six singularity types are re-defined in a way relevant to redundant mechanisms. In Section 4, we study the interdependence of the singularity types and prove classification theorems for redundant mechanisms.

2. THE VELOCITY SPACE AND THE VELOCITY EQUATION

For the present study of the singular configurations of an *arbitrary kinematic chain* it can be assumed that all N kinematic pairs have 1 dof. The *full-cycle mobility* of the mechanism (Hunt 1978) is denoted by n. Only n_I of the N joints (the input joints) are *active*, i.e., their joint parameters can be actively changed, while the remaining $N - n_I$ joints are *passive*. The dimension of the output space of the mechanism will be denoted by n_O. The n_I active joint velocities will be referred to as *input*, and the n_O differential output parameters (the output velocities), specifying the instantaneous motion of the output link, as *output*.

For an arbitrary mechanism, we have $n_I \geq n \geq n_O$. The mechanism is non-redundant if $n_I = n = n_O$. *Redundancy* is present when the dimensions of the input and output space are not equal and, therefore, either $n > n_O$ or $n_I > n$. The first inequality defines *kinematic redundancy.* If $n_I > n$ we will say that a *dynamic redundancy* is present. This phenomenon is often referred to as *actuation redundancy* in the literature. In this paper, it is be assumed that $n_I \geq n \geq n_O$ and, unless the opposite is explicitly specified, all propositions are valid for redundant and non-redundant mechanisms alike. Moreover, the classification framework, proposed herein, when applied to non-redundant mechanisms, must be equivalent to the respective results presented in (Zlatanov et al., 1995).

A *velocity vector* is defined as an $(N + n_O)$-tuple, $\boldsymbol{m} = [\boldsymbol{T}^T, \boldsymbol{\Omega}^{a\,T}, \boldsymbol{\Omega}^{p\,T}]^T$, where $\boldsymbol{T}$, $\boldsymbol{\Omega}^a$ and $\boldsymbol{\Omega}^p$ are the column vectors (with dimensions n_O, n_I and $N - n_I$) of the output velocities, active-joint velocities, and the the passive-joint velocities, respectively. The velocity vectors are elements of an $(N + n_O)$-dimensional vector space $\mathcal{V}$, referred to as the *velocity space*. $\mathcal{V}$ has the structure $\mathcal{V} = O \oplus I \oplus \mathcal{P}$, where O is the n_O-dimensional output-coordinates subspace of $\mathcal{V}$, I is the n_I-dimensional input-coordinates subspace of $\mathcal{V}$, and $\mathcal{P}$ is the $(N - n)$-dimensional passive-joint-velocity-coordinates subspace of $\mathcal{V}$.

A velocity vector $\boldsymbol{m}$, which represents a feasible motion for the mechanism in a given configuration, $\boldsymbol{q}$, will be referred to as a *motion vector* for $\boldsymbol{q}$. The set of all motion vectors for $\boldsymbol{q}$, $\mathcal{M}_q$, a linear subspace of $\mathcal{V}$ with $\dim \mathcal{M}_q = n_q$, is the *motion space*. All properties of the instantaneous kinematics of the mechanism are determined by the orientation of $\mathcal{M}_q$ in $\mathcal{V}$.

For any configuration, $\boldsymbol{q}$, there exists a matrix with dimensions $(N - n + n_O) \times (N + n_O)$, denoted $\boldsymbol{L}(\boldsymbol{q})$, such that $\boldsymbol{m}$ is a feasible instantaneous motion of the mechanism if and only if:

$$\boldsymbol{L}(\boldsymbol{q})\boldsymbol{m} = 0. \qquad (1)$$

Equation (1) will be referred to as the *velocity equation* for $\boldsymbol{q}$. The existence of the matrix $\boldsymbol{L}$ is a corollary of the fact that the feasible arrays of joint velocities of a mechanism are given as the solution vectors of a system of $N - n$ linear equations (Freudenstein 1962, Davies 1981).

$\mathcal{M}_q$ is, in fact, the solution space of the velocity equation. Two maps, $\boldsymbol{p}_I : \mathcal{M}_q \rightarrow I$ and $\boldsymbol{p}_O : \mathcal{M}_q \rightarrow O$, are defined as the restrictions on $\mathcal{M}_q$ of the projections which map $\mathcal{V}$ onto I and O. $\boldsymbol{p}_I$ and $\boldsymbol{p}_O$ map any motion vector into the vector of its input or output, respectively. Their ranks, i.e., the dimensions of their image spaces, are r_I and r_O, respectively. The dimensions of their null-spaces will be denoted by d_I and d_O, respectively ($d_I = n_q - r_I$ and $d_O = n_q - r_O$). Additionally, we introduce the notation d_{IO}, defined as:

$$d_{IO} = \dim(\operatorname{Ker} \boldsymbol{p}_I \cap \operatorname{Ker} \boldsymbol{p}_O).$$

Note that, the maps $\boldsymbol{p}_I$ and $\boldsymbol{p}_O$ (and their ranks) are dependent on the configuration $\boldsymbol{q}$.

3. SINGULARITY AND SINGULARITY TYPES FOR REDUNDANT MECHANISMS

In this section we give new, more general definitions for the six singularity types, first defined in Zlatanov et al. (1995). First, we examine how the instantaneous formulation of the definition of singularity changes when redundancy is possible.

3.1. Singularity

For non-redundant mechanisms, nonsingularity is defined by the solvability of the forward and inverse kinematics. A configuration is nonsingular when the entire instantaneous motion of the mechanism can be recovered from the input as well as from the output. This is equivalent to the requirement that both $\boldsymbol{p}_I$ and $\boldsymbol{p}_O$ are bijective. This, however, is not possible if the mechanism is redundant and $n_I \neq n_O$. Instead, the following definition can be used.

1. ***Definition.*** *A configuration, $\boldsymbol{q}$, is* **nonsingular**, *when*

$$r_I = n = n_q \ \text{ and } \ r_O = n_O \,.$$

Otherwise, $\boldsymbol{q}$ is said to be a **singular** *configuration (or a singularity).*

2. ***Remark.*** Definition 1 imposes three conditions for nonsingularity. Firstly, the global mobility, n, must be equal to the dimension of the motion space, n_q, also referred to as the instantaneous or transitory mobility (Hunt, 1978). If this condition is violated, $\boldsymbol{q}$ is a singular point of the configuration space of the mechanism. The other two equalities, $r_I = n_q$ and $r_O = n_O$, ensure that the maps $\boldsymbol{p}_I$ and $\boldsymbol{p}_O$ are nonsingular.

In the following six sub-sections, the six singularity types are defined and interpreted, using the velocity equation and the motion-space projections. Each type is illustrated with an example, and the formulation with redundancy is compared to the non-redundant case (Zlatanov et al., 1995).

3.2. Redundant Input

3. ***Definition.*** *A configuration is a singularity of* **redundant input** *(RI) type, if there exist at least $n - n_O + 1$ linearly independent input vectors, $\boldsymbol{\Omega}_1^a, \ldots, \boldsymbol{\Omega}_{n-n_O+1}^a$, such that each of them satisfies the velocity equation for a zero output, $\boldsymbol{T} = 0$ (and some vector of passive-joint velocities, $\boldsymbol{\Omega}^p$), i.e., for every i, the following equation is satisfied for some $\boldsymbol{\Omega}^p$:*

$$\boldsymbol{L} \begin{bmatrix} 0 \\ \boldsymbol{\Omega}_i^a \\ \boldsymbol{\Omega}^p \end{bmatrix} = 0. \qquad (2)$$

4. ***Proposition.*** *A necessary and sufficient condition for an RI-type singularity is the inequality:*

$$\dim(\operatorname{Ker} \boldsymbol{p}_O) - \dim(\operatorname{Ker} \boldsymbol{p}_I \cap \operatorname{Ker} \boldsymbol{p}_O) > n - n_O,$$

or equivalently,

$$d_O - d_{IO} > n - n_O. \qquad (3)$$

Proof. The condition (3) is satisfied, if and only if there is a subspace of $\operatorname{Ker} \boldsymbol{p}_O$, $\mathcal{S}$, of dimension greater than $n - n_O$, which is complementary to $\operatorname{Ker} \boldsymbol{p}_I \cap \operatorname{Ker} \boldsymbol{p}_O$. In other words, $\mathcal{S}$ is such that

$$\mathcal{S} + (\operatorname{Ker} \boldsymbol{p}_I \cap \operatorname{Ker} \boldsymbol{p}_O) = \operatorname{Ker} \boldsymbol{p}_O$$

and

$$\mathcal{S} \cap (\text{Ker}\, \boldsymbol{p}_I \cap \text{Ker}\, \boldsymbol{p}_O) = 0,$$

i.e.,

$$\mathcal{S} \oplus (\text{Ker}\, \boldsymbol{p}_I \cap \text{Ker}\, \boldsymbol{p}_O) = \text{Ker}\, \boldsymbol{p}_O.$$

This observation proves the Proposition, since any basis of $\mathcal{S}$ provides the "redundant-input" vectors required by Definition 3, while when the existence of such vectors is given, their linear envelope gives the subspace $\mathcal{S}$ needed to establish Eq. (3). □

5. ***Remark.*** Proposition 4 can be used to show the correctness of Definition 3, i.e., to show that the RI-type configurations (as defined by Definition 3) are indeed singular (according to Definition 1). Similarly, we note here that the definitions of the other five singularity types (defined in the following subsections) can be showed to be compatible with Definition 1.

6. ***Remark.*** A comparison of Definition 3 with the corresponding definition for non-redundant mechanisms (Zlatanov et al., 1995) shows that the conditions for an RI-type singularity have been modified in order to include mechanisms with redundancy. While in the non-redundant case an RI-type singularity is associated with the existence of at least one motion with zero output and nonzero input, in the redundant case, a whole space of such motions, with dimension larger than the degree of kinematic redundancy, $n - n_O$, is required.

The reason for this difference is that when $n > n_O$, the existence of only one motion with zero output is no longer a sufficient condition for the occurrence of singularity. Indeed, fixing the output to zero removes only n_O freedoms, which is not sufficient to immobilize a mechanism with mobility higher than n_O. Thus, even in a nonsingular configuration $n - n_O$ "redundant-input" motions are expected to exist.

7. ***Example.*** As a simple example of an RI-type singularity for a redundant mechanism, let us consider the configuration in Figure 1.

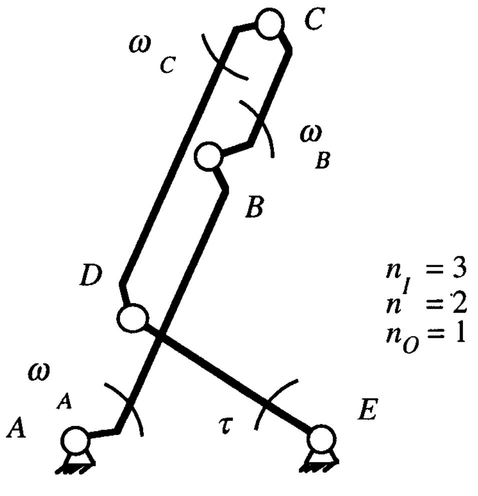

Figure 1. A singular configuration of class (RI, IO).

This is a five-bar linkage with three inputs and a single output. The input velocities are the joint velocities at points A, B and C. The output velocity is the angular velocity of the link ED. The general mobility of the mechanism is two.

In the configuration shown in Figure 1, the output velocity is always zero and the remainder of the linkage forms a flattened four-bar linkage with a mobility of two. The maximum number of linearly independent inputs equals two, which is greater than $n - n_O$. Indeed, the two "redundant-input" vectors can be chosen by fixing, respectively, the joint velocities at A and at B to be zero. Checking Equation (3), the mobility with fixed input *and* output, d_{IO}, is zero, while the mobility with fixed output is $d_O = 2$. The difference, $d_O - d_{IO} = 2$, is greater than the degree of kinematic redundancy, $n - n_O = 1$.

3.3. Redundant Output

8. ***Definition.*** *A configuration is said to be a singularity of* **redundant output** *(RO) type, if there exist a nonzero output, $\boldsymbol{T} \neq 0$, and a vector of passive-joint velocities, $\boldsymbol{\Omega}^p$, which satisfy the velocity equation for a zero-input, $\boldsymbol{\Omega}^a = 0$:*

$$L \begin{bmatrix} \boldsymbol{T} \\ 0 \\ \boldsymbol{\Omega}^p \end{bmatrix} = 0. \tag{4}$$

9. ***Proposition.*** *A necessary and sufficient condition for an RO-type singularity is the inequality:*

$$\dim(\text{Ker}\, \boldsymbol{p}_I) - \dim(\text{Ker}\, \boldsymbol{p}_I \cap \text{Ker}\, \boldsymbol{p}_O) > 0,$$

or equivalently,

$$d_I - d_{IO} > 0. \tag{5}$$

Proof. Equation (5) is equivalent to $\text{Ker}\, \boldsymbol{p}_I - \text{Ker}\, \boldsymbol{p}_O \neq \emptyset$. This condition, stating that there are motion vectors with zero input and nonzero output, is equivalent to the requirement of Definition 8. □

10. ***Remark.*** Comparing the above formulation of the RO-type with the definition for the non-redundant case (Zlatanov et al., 1995), we note that, for this singularity type, the definition does not change when redundancy is introduced. As a result, for redundant mechanisms the RO-type definition does not mirror the RI-type definition as closely as in the non-redundant case. As we shall see later, this leads to a loss of the input-output symmetry in the redundant-mechanism singularity classification.

The reason for keeping the same RO-type definition for both redundant and non-redundant mechanisms is that it ensures that the configuration is singular even when the mechanism is redundant. In fact, for redundant mechanisms the requirement for an RO-type configuration is even harder to satisfy. Indeed, for dynamically-redundant mechanisms ($n_I > n$), d_I will be smaller, since when the inputs are fixed to be zero, a greater number of dof may be lost. If the mechanism has nonzero mobility when the inputs are zero (i.e., $d_I > 0$), then a kinematically-redundant mechanism will be more likely to have a higher d_{IO}, since it has fewer outputs. For example, one can note that a five-bar mechanism with three inputs (Figure 1, Example 7) can have no RO-type singularities. In any configuration, if the first three joints are locked no link can move.

11. ***Example.*** The mechanism shown in Figure 2 has: $n_I = 3$, $n = 2$, $n_O = 1$. The input velocities are the joint velocities at joints *A*, *E* and *F*. The output is the motion of the slider *G*.

Figure 2 depicts an RO-type singular configuration. The configuration is such that the points *B*, *C*, *D*, *F*, *G* are aligned and this line is perpendicular to the prismatic-joint axis. It can be seen that even when the joint velocities at points *A*, *E* and *F* are zero, the output slider can still move. It can be shown that, in this configuration, $d_I = 1$ and $d_{IO} = 0$. Therefore, Equation (5) is satisfied.

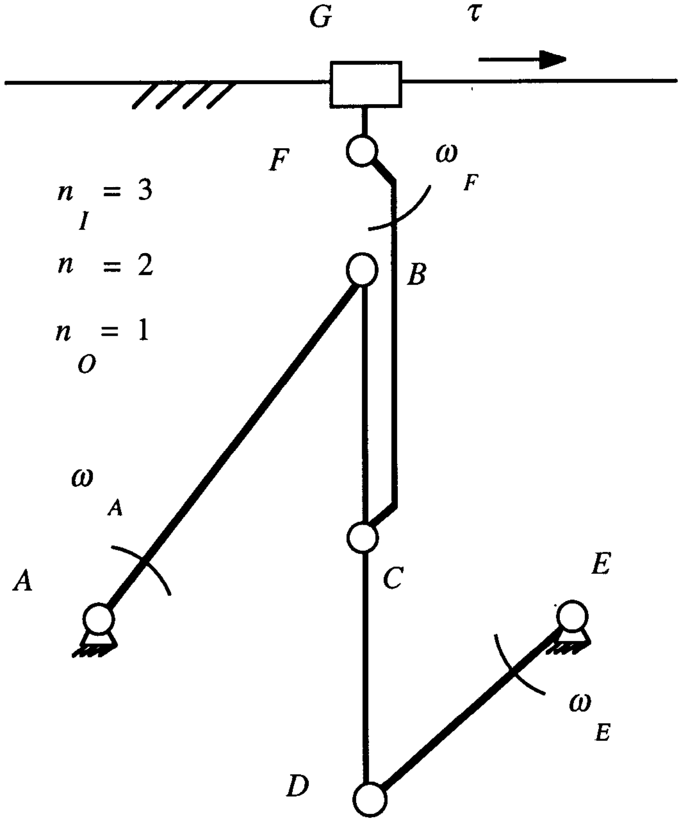

Figure 2. A singularity of class (RO, II).

3.4. Impossible Input

12. ***Definition.*** *A configuration is a singularity of* **impossible input** *(II) type, if there exists a subspace of* I, $\mathcal{S}$, *of dimension higher than the degree of dynamic redundancy,* $n_I - n$, *such that for every vector* $\Omega^a \neq 0$ *in* $\mathcal{S}$ *the velocity equation cannot be satisfied for any choice of* $\mathbf{T}$ *and* Ω^p.

13. ***Proposition.*** *A necessary and sufficient condition for an II-type singularity is the inequality:*

$$r_I < n,$$

or equivalently,

$$n_q - n < d_I. \qquad (6)$$

Proof. First, we note that $r_I < n$ is equivalent to Equation (6) because of $r_I = n_q - d_I$.

The inequality $r_I < n$ holds if and only if there is a subspace *of* I, $\mathcal{S}$, with dimension $n_I - n$ or more, such that $\mathcal{S} \oplus \operatorname{Im} \mathbf{p}_I = I$. Since none of the nonzero elements of $\mathcal{S}$ is in $\operatorname{Im} \mathbf{p}_I$, $\mathcal{S}$ satisfies the requirements of Definition 12. □

14. ***Remark.*** Similarly to the definition of the RI type, the definition of II-type singularities is different in the redundant and non-redundant cases. A comparison of Definition 12 and the corresponding definition by Zlatanov et al. (1995) shows that in the redundant case the definition is more restrictive. It is no longer sufficient to establish the existence of a single "impossible-input" vector, rather an "impossible-input" subspace must be present. This means that II-type configurations are more "rare" for dynamically-redundant mechanisms. For example, the 5-bar mechanism with three inputs shown in Figure 1 cannot have an II-type singularity since d_I is obviously zero (and therefore, by Equation (6), an II-type singularity cannot be present).

As in the case of the RI-type definition (see Remark 6), the requirement of the II-type non-redundant-case definition does not guarantee that the configuration is singular, moreover, for dynamically-redundant mechanisms this condition is satisfied in all configurations. Indeed, there can be no more than *n* linearly independent feasible inputs, and when $n_I > n$, there must exist input vectors that are not feasible for the mechanism.

Definition 12 ensures that the configuration is singular by requiring that II-type configurations have "more" impossible inputs than a nonsingular configuration.

15. ***Example.*** Consider again the configuration shown in Figure 2, which was used as an example for an RO-type singularity (Example 11). It is an II-type singularity as well. Indeed, in this configuration, $d_I = 1$ and $n = n_q = 2$. According to Equation (6), the configuration is an II-type singularity.

3.5. Impossible Output

16. ***Definition.*** *A configuration is a singularity of* **impossible output** *(IO) type, if there exists a vector* $\mathbf{T}$ *for which the velocity equation cannot be satisfied for any combination of* Ω^a *and* Ω^p.

17. ***Proposition.*** *A necessary and sufficient condition for an IO-type singularity is the inequality:*

$$r_O < n_O,$$

or equivalently,

$$n_q - n_O < d_O. \qquad (7)$$

Proof. First, $r_O < n_O$ is equivalent to Equation (7) because of $r_O = n_q - d_O$.

The inequality $r_O < n_O$ holds, if and only if there is at least one output vector, which corresponds to no feasible instantaneous motion, i.e., it is equivalent to Definition 16. ☐

18. Remark. Comparing the above formulation of the IO-type with the corresponding definition for non-redundant mechanisms, we note that for this singularity type the definition does not change when redundancy is introduced. As a result, for redundant mechanisms the IO-type definition does not mirror the II-type definition as closely as in the non-redundant case. This leads to a loss of the input-output symmetry in the redundant-mechanism singularity classification. Keeping the non-redundant-case definition is possible since it ensures that the configuration is singular even when the mechanism is redundant.

19. Example. The configuration shown in Figure 1, and discussed in Example 7, is an IO-type singularity. It can be seen that the joint at point E is locked. (Indeed, point D cannot have a velocity component parallel to the line along A, B and C.) Also, since $d_O = 2$, $n_q = 2$ and $n_O = 1$ (see Example 7), Equation (7) is satisfied.

3.6. Increased Instantaneous Mobility

20. Definition. *A configuration is a singularity of* **increased instantaneous mobility** *(IIM) type, if* L *is singular, i.e.,* $\operatorname{rank} L < N - n + n_O$.

21. Proposition. *An IIM-type singularity is present, if and only if* $n < n_q$.

Proof. The solution space of the velocity equation is the motion space, $\mathcal{M}_q$, which has a dimension of n_q. Therefore, $\operatorname{rank} L = N + n_O - n_q$. Hence, $\operatorname{rank} L < N - n + n_O$ is equivalent to $n < n_q$. ☐

22. Remark. An IIM-type singularity occurs if and only if the configuration is a singular point of the configuration space of the mechanism, D. Therefore, it does not depend on the choice of the active joints or the output link. IIM is a property of the kinematic chain and is therefore not influenced by redundancy. Thus, the configurations of non-redundant mechanisms, that have been shown to be IIM-type singularities, can be used as examples for the redundant case. It suffices to assume that some of the passive joints are active or redefine the output. This cannot be done for the other singularity types, since they are affected by the way the input and output are chosen.

23. Example. Consider, for example, the configuration shown in Figure 3. The mechanism is similar to the one shown in Figure 1, i.e., a five-bar linkage with three input joints and a single output.

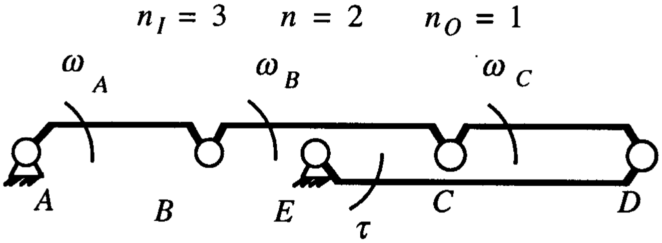

Figure 3. An (RI, IIM)-class singular configuration.

This flattened configuration is an IIM-type singularity for any five-bar mechanism, regardless of the choice or number of input joints or output parameters. Since in all cases we have $2 = n < n_q = 3$, the condition of Proposition 21 is satisfied. For the mechanism in Figure 3, we also have: $n_I = 3$, $n = 2$, $n_O = 1$, and $d_{IO} = 0$, $d_I = 0$, $d_O = 2$. Applying the singularity-type definitions in this section (and Propositions 4, 9, 13 and 17) we conclude that the configuration shown also belongs to the RI type but does not belong to types RO, IO, or II (it does not belong to the RPM-type either, as can be seen by applying Definition 24 below). This indicates that the combinations of singularity types for redundant mechanisms obey rules different from the ones for non-redundant mechanisms, published by Zlatanov et al. (1995). (For non-redundant mechanisms, a configuration cannot belong only to the IIM and RI types).

3.7. Redundant Passive Mobility

24. Definition. *A configuration is a singularity of* **redundant passive motion** *(RPM) type, if there exists a nonzero passive-joint-velocity vector,* $\Omega^P \neq 0$, *which satisfies the velocity equation for a zero input and a zero output, i.e.,*

$$L \begin{bmatrix} 0 \\ 0 \\ \Omega^P \end{bmatrix} = 0. \qquad (8)$$

25. Proposition. *An RPM-type singularity is present, if and only if*

$$d_{IO} > 0. \qquad (9)$$

Proof. The inequality (9) holds, if and only if the intersection $\operatorname{Ker} \boldsymbol{p}_I \cap \operatorname{Ker} \boldsymbol{p}_O$ has a dimension of at least one. Therefore, there is a nonzero motion vector, which is mapped into zero by both $\boldsymbol{p}_I$ and $\boldsymbol{p}_O$, i.e., a nonzero instantaneous motion with zero input and zero output. ☐

26. Remark. Definition 24 is identical with the definition in the non-redundant case. The definition requirement ensures that the configuration is singular, so there is no need to modify the definition for the redundant case. In general, the chances for the existence of an RPM-type singularity improve when the combined total of the inputs and outputs is decreased, and vice versa. Therefore, dynamic and kinematic redundancy have a different effect on RPM-type singularities. A smaller number of outputs facilitates the occurrence of RPM-type singularities, while an increase in the number of active joints makes it more difficult for RPM-type configurations to occur.

27. ***Example.*** As an illustration of the RPM singularity type, we use another configuration of the kinematic chain described in Examples 11 and 15. The configuration considered here, shown in Figure 4, is very similar to the one in Figure 2, but this time the point G is not aligned with B, C, D, F. Unlike Examples 11 and 15, it is assumed herein that the mechanism has only two active joints, namely A and E.

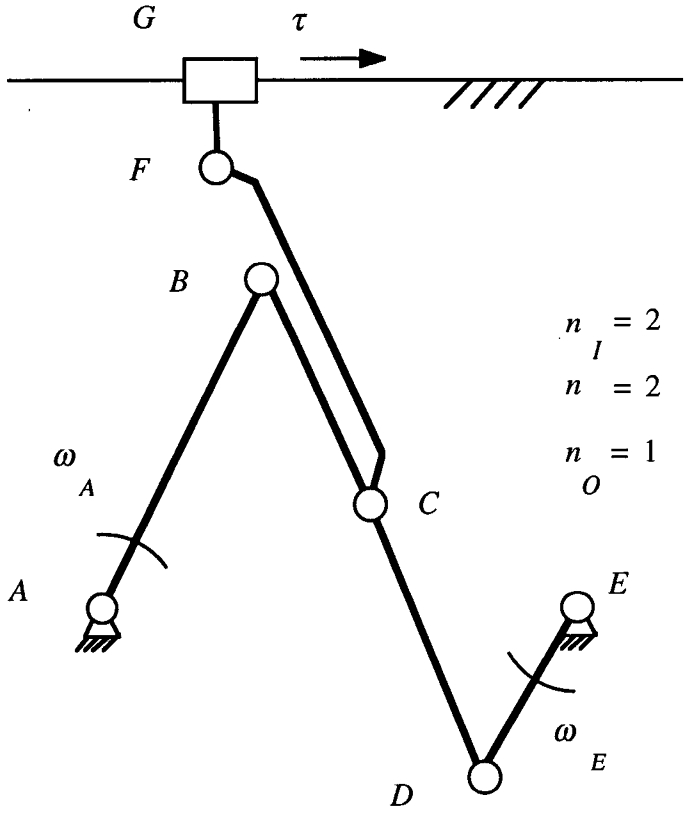

Figure 4. A singular configuration of class (RPM, II).

It is verified that $d_{IO} = d_I = d_O = 1$ and $n_q = n = 2$. Then, it is easy to check that the only singularity types that are present are of the RPM and II types. We note that the singularity class (RPM, II) is not among the ones occurring in non-redundant mechanisms. The passive motion, which can take place with the input and output equal to zero, occurs with an instantaneous motion of point C along a line normal to the line BD. It can be noted that, if the joint F were active as well, the configuration would no longer be of the RPM type.

Finally, we summarize the singularity-type definitions in the present section with Table 1.

Type	Condition
RI	$d_O - d_{IO} > n - n_O$
RO	$d_I - d_{IO} > 0$
II	$d_I > n_q - n$
IO	$d_O > n_q - n_O$
IIM	$n_q - n > 0$
RPM	$d_{IO} > 0$

Table 1. Definitions of the singularity types for mechanisms with redundancy.

4. CLASSIFICATION OF SINGULARITIES

4.1. Combinations of Singularity Types

As a first step towards a general classification of the singularities of arbitrary mechanisms, we study the interdependence of the six singularity types. A singular configuration never belongs to a single singularity type but rather to a *combination of singularity types*. The following proposition provides the rules which will allow us to distinguish the possible combinations from the impossible ones.

28. ***Proposition***

(i) $q \in \{\mathrm{RI}\} \Rightarrow q \in \{\mathrm{IO}\}$ or $q \in \{\mathrm{IIM}\}$,
(ii) $q \in \{\mathrm{RO}\} \Rightarrow q \in \{\mathrm{II}\}$ or $q \in \{\mathrm{IIM}\}$,
(iii) $q \in \{\mathrm{II}\} \Rightarrow q \in \{\mathrm{RO}\}$ *or* $q \in \{\mathrm{RPM}\}$,
(iv) $q \in \{\mathrm{IO}\} \Rightarrow q \in \{\mathrm{RI}\}$ *or* $q \in \{\mathrm{RPM}\}$,
(v) $q \in \{\mathrm{RPM}\} \Rightarrow q \in \{\mathrm{II}\}$ *or* $q \in \{\mathrm{IIM}\}$,
(vi) $q \in \{\mathrm{IIM}\} \Rightarrow q \in \{\mathrm{RI}\}$ *or* $q \in \{\mathrm{RPM}\}$,
(vii) $q \in \{\mathrm{RO}\} \Rightarrow q \in \{\mathrm{II}\}$ *or* $q \in \{\mathrm{RI}\}$,
(viii) $q \in \{\mathrm{IO}\} \Rightarrow q \in \{\mathrm{II}\}$ *or* $q \in \{\mathrm{RI}\}$,
(ix) $q \in \{\mathrm{II}\}$ *and* $q \in \{\mathrm{RI}\} \Rightarrow q \in \{\mathrm{IO}\}$ *or* $q \in \{\mathrm{RO}\}$.

Proof

(i) An RI-type singularity is given. Assume that there is no IIM-type singularity. Thus, we have $d_O - d_{IO} > n - n_O$ and $n = n_q$. Therefore,

$$d_O \geq d_O - d_{IO} > n - n_O = n_q - n_O.$$

Then, the inequality $d_O > n_q - n_O$ implies an IO-type singularity (Proposition 17).

(ii) If q is an RO-type singularity, then $d_I - d_{IO} > 0$. We assume that q is not an IIM-type singularity, i.e., $n = n_q$. Therefore,

$$d_I \geq d_I - d_{IO} > 0 = n_q - n.$$

However, $d_I > n_q - n$, implies an II-type singularity.

(iii) $q \in \{\mathrm{II}\}$ implies $d_I > n_q - n$. Let us assume that the configuration is not an RPM-type singularity, or, equivalently, that $d_{IO} = 0$. Therefore , we can write:

$$d_I - d_{IO} = d_I > n_q - n \geq 0,$$

which, according to Proposition 9, implies that q is an RO-type singularity.

(iv) By Proposition 17, the condition for an IO-type singularity is $d_O > n_q - n_O$. It is assumed that the singularity is not of the RPM type, i.e., $d_{IO} = 0$. Thus, we have

$$d_O - d_{IO} = d_O > n_q - n_O \ge n - n_O,$$

i.e., $d_O - d_{IO} > n - n_O$, which is the condition defining an RI-type singularity.

(v) An RPM-type singularity is given, thus, $d_{IO} > 0$. Assume q is not an IIM-type singularity, i.e., $n_q = n$. Since d_I is always at least as large as d_{IO}, it follows that:

$$d_I \ge d_{IO} > 0 = n_q - n.$$

This proves that $d_I > n_q - n$, i.e., the configuration is an II-type singularity.

(vi) An IIM-type singularity is equivalent to $n_q > n$. We assume $d_{IO} = 0$, (i.e., that the configuration is not an RPM-type singularity). Then,

$$d_O - d_{IO} = d_O = n_q - r_O \ge n_q - n_O > n - n_O$$

Above, we have used $n_O \ge r_O$ (the rank of a map cannot exceed the dimension of the target space). Thus, the inequality $d_O - d_{IO} > n - n_O$ is obtained and this ensures the presence of an RI-type singularity at q.

(vii) It is given that $d_I > d_{IO}$ (RO-type singularity). Let us assume that the configuration does not belong to the II type, hence $d_I \le n_q - n$. Then, we can write:

$$d_O - d_{IO} > d_O - d_I \ge d_O - (n_q - n) = n - (n_q - d_O) = n - r_O \ge n - n_O.$$

Above, the first two inequalities follow from $d_I > d_{IO}$ and $d_I \le n_q - n$, respectively. The last inequality uses $r_O \le n_O$. It is established that $d_O - d_{IO} > n - n_O$, i.e., the configuration belongs to the RI type.

(viii) It is known that q is an IO-type singularity. This requires $d_O > n_q - n_O$. If the configuration is not an II-type singularity as well, then $d_I \le n_q - n$. We have:

$$d_O - d_{IO} > (n_q - n_O) - d_{IO} \ge (n_q - n_O) - d_I \ge (n_q - n_O) - (n_q - n) = n - n_O.$$

The second inequality uses $d_I \ge d_{IO}$. Once again, we obtain $d_O - d_{IO} > n - n_O$, hence the configuration belongs to the RI type.

(ix) It is given that q belongs to both the II and RI singularity types. This implies two inequalities, $d_I > n_q - n$ and $d_O - d_{IO} > n - n_O$, respectively. If we assume that q is not of the IO type, we must also have $d_O = n_q - n_O$. Using these conditions, the following inequalities can be written:

$$d_I - d_{IO} > (n_q - n) - d_{IO} = n_q - n + n_O - n_O - d_{IO} = (n_q - n_O) - d_{IO} - (n - n_O) = d_O - d_{IO} - (n - n_O) > 0.$$

This proves $d_I > d_{IO}$ and, therefore, the occurrence of an RO-type singularity. □

29. *Remark.* The above proposition can be compared to an analogous result for non-redundant mechanisms (Zlatanov et al., 1995, Proposition 4). One can note that only six of the statements in the previous result could be proven for redundant mechanisms. These are statements (i)–(iv), (vii) and (vii) in Proposition 28. The remaining three statements in Proposition 28, namely (v), (vi), and (ix) are weaker versions of the corresponding statements for non-redundant mechanisms. In fact, we already presented proof that stronger statements, identical with the ones in Zlatanov et al. (1995) would be incorrect. In Example 23, a configuration which belongs only to the types IIM and RI was shown (Figure 3). This example disproves the statements:

$$q \in \{\text{IIM}\} \Rightarrow (q \in \{\text{RI}\} \textit{ and } q \in \{\text{RO}\}) \textit{ or } q \in \{\text{RPM}\},$$
$$q \in \{\text{RI}\} \Rightarrow q \in \{\text{IO}\} \textit{ or } q \in \{\text{RO}\},$$

which are true for non-redundant mechanisms. Another configuration, introduced with Example 27 (Figure 4), belongs to the RPM and II types and to no other type. Therefore, the following two propositions (valid in the non-redundant case) cannot be generalized for redundant mechanisms:

$$q \in \{\text{RPM}\} \Rightarrow q \in \{\text{II}\} \textit{ or } q \in \{\text{IIM}\},$$
$$q \in \{\text{II}\} \Rightarrow q \in \{\text{IO}\} \textit{ or } q \in \{\text{RO}\}.$$

4.2. General Singularity Classification

The goal of this section is to classify the set of all singularities of all mechanisms. This set is divided into classes, using as a criterion the combination of singularity types to which a configuration belongs. More precisely, two configurations are considered equivalent (are in the same singularity class), when they belong to exactly the same singularity types. This is a relation of equivalence which divides the set of all singularities into non-intersecting classes. In this sub-section, we identify the combinations for which there exist configurations and which, therefore, correspond to non-empty singularity classes. By listing all non-empty singularity classes, we develop a comprehensive classification.

30. *Proposition.* *Let q be a singular configuration. Then,*

(i) q belongs to at least one of the types IO, II, and IIM.

(ii) q belongs to at least one of the types RO, RI, and RPM.

Proof.

(i) According to Definition 1, a configuration, q, is singular in (at least) one of three cases: $n \ne n_q$, which is the condition for the IIM type; $r_O < n_O$, equivalent to an IO-type singularity; and $r_I < n_q = n$ implying an II-type singularity.

(ii) Follows from (i) and Proposition 28 (iii), (iv) and (vi). □

31. *Theorem.* *Let S be an arbitrary combination of some of the six singularity types. There exists a mechanism with a configuration, q, which belongs to all types in S and to no other types, if and only if S is marked with "Y", "K" or "D" in Table 2.*

	IO	II	IO *and* II	IIM	IO *and* IIM	II *and* IIM	IO *and* II *and* IIM
RI	**Y**			**D**	**D**		
RO		**Y**					
RI*and*RO			**Y**	**Y**	**Y**	**Y**	**Y**
RPM		**K**	**Y**	**Y**		**K**	**Y**
RI*and*RPM			**Y**	**D**	**Y**		**Y**
RO*and*RPM		**K**	**Y**			**Y**	**Y**
RI*and*RO*and*RPM			**Y**	**Y**	**Y**	**Y**	**Y**

Table 2. Possible combinations of singularity types for redundant mechanisms.

	IO	II	IO *and* II	IIM	IO *and* IIM	II *and* IIM	IO *and* II *and* IIM
RI	Y	*(iii)*	*(iii)*	Y	Y	*(iii)*	*(iii)*
RO	*(ii)*	Y	*(iv)*	*(vi)*	*(iv)*	*(vi)*	*(iv)*
RI*and*RO	*(ii)*	*(i)*	Y	Y	Y	Y	Y
RPM	*(v)*	Y	Y	Y	*(viii)*	Y	Y
RI*and*RPM	*(v)*	*(i)*	Y	Y	Y	*(ix)*	Y
RO*and*RPM	*(ii)*	Y	Y	*(vii)*	*(viii)*	Y	Y
RI*and*RO*and*RPM	*(ii)*	*(i)*	Y	Y	Y	Y	Y

Table 3. Impossible combinations of singularity types for redundant mechanisms.

Proof. To prove the theorem, we need to establish that (i) all combinations not marked in the table can never occur and (ii) there exist mechanisms and configurations with the marked singularity-type combinations.

(i) There are six singularity types and therefore there are $2^6 = 64$ combinations. Proposition 30 implies that it is sufficient to consider the ones that include at least one I-type and one R-type. These combinations are represented by the 49 cells of Table 2. The cell in the i-th row and j-th column of the table corresponds to a combination of all singularity types listed to the left of the i-th row and on top of the j-th column.

We must show that the combinations corresponding to blank cells of the table are impossible. This is proven with the help of Proposition 28. Each of the 22 empty cells represents a combination of singularity types which, if it occurred in some configuration, would violate (at least) one of the statements of Proposition 28. Table 3 illustrates which statement each blank cell violates.

(ii) It is sufficient to give an example for each of the 27 remaining combinations. Lack of space prevents us to present a comprehensive set of examples. All necessary configurations are given by Zlatanov (1998). In the present paper, we illustrate the singularity classes that cannot occur for non-redundant mechanisms. Two such classes, namely (RI, IIM) and (RPM, IO), were already obtained in Example 23 (Figure 3) and Example 27 (Figure 4), respectively. The four remaining classes are given with the following four examples.

32. ***Example.*** Let us consider the five-bar linkage in its flattened configuration shown in Figure 5. The output is the position of point C, while there are three input joints at A, B and E.

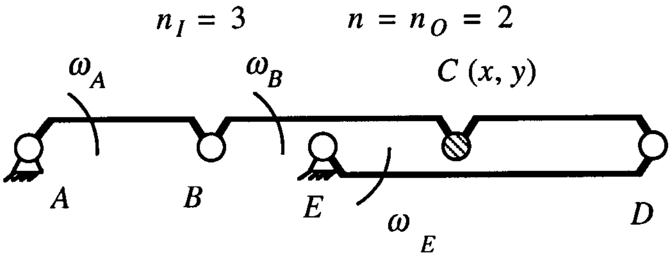

Figure 5. An (RI, IO, IIM)-class singularity.

Therefore, $n_I = 3$, $n = n_O = 2$. It is found that $d_{IO} = 0$, $d_I = 0$, $d_O = 2$, $n_q = 3$. This implies a singular configuration of the (RI, IO, IIM) class.

33. ***Example.*** The configuration shown in Figure 6 is the same as in Figure 2 (discussed in Example 11). However, in the present example, the joint F is passive, thus $n_I = n = 2$. It is found that $d_{IO} = 1$, $d_I = 2$, $d_O = 1$ and $n_q = 2$. According to the defining conditions given in Table 1, the configuration is of the singularity-type combination (RPM, RO, II).

34. ***Example.*** We consider the configuration shown in Figure 7. The mechanism is similar to the one in Figure 2, however, it has a four-bar linkage $ABCD$ (rather than a five-bar) connected to the slider by the link CE. As a result, the mobility is 1 rather than 2. The output is the motion of the slider, the input joints are B and E. Here, all rotary joints are aligned and two of them, A and C, coincide. Thus, $n_I = 2$, $n = n_O = 1$. We establish that $d_{IO} = 1$, $d_I = 1$, $d_O = 2$, $n_q = 3$, which implies a singularity-type combination (RI, RPM, IIM).

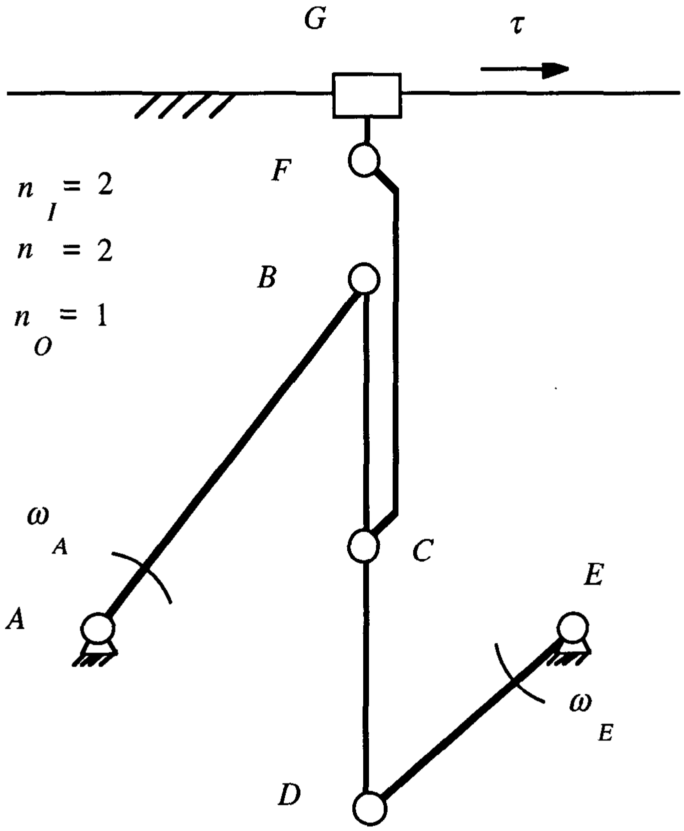

Figure 6. An (RO, RPM, II)-class singular configuration.

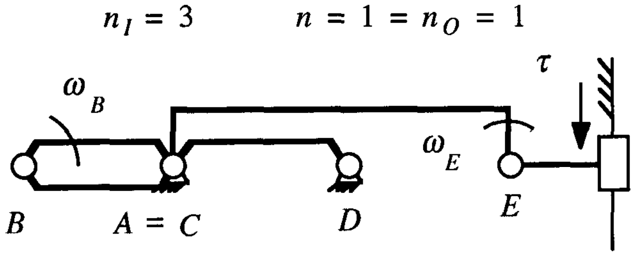

Figure 7. A singular configuration of class (RI, RPM, IIM).

35. ***Example.*** Finally, let us consider the mechanism shown in Figure 8.

The output is the velocity of the slider, i.e., $n_O = 1$. The joints at points A, F and K are active and thus $n_I = n = 3$. In the configuration shown in Figure 9, it is not difficult to see that $d_{IO} = 3$, $d_I = 3$, $d_O = 4$ and $n_q = 5$. This translates into a singularity of the (RPM, II, IIM)-class.

With this the proof of Theorem 31 is completed. ☐

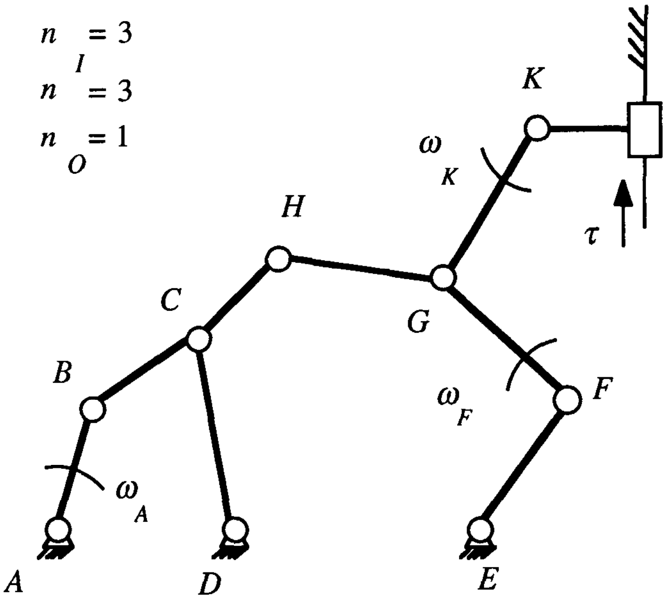

Figure 8. A 3-dof redundant planar mechanism.

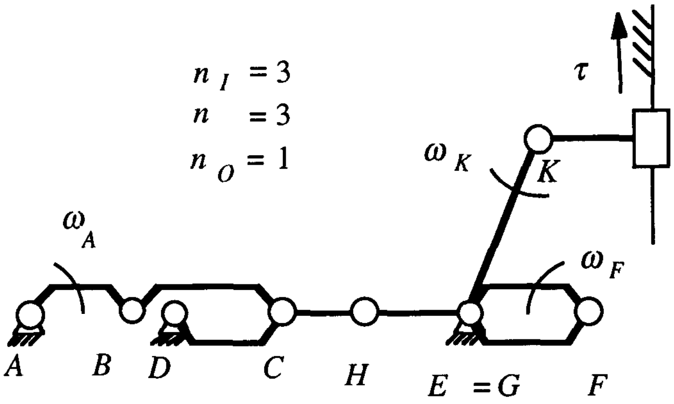

Figure 9. An (RPM, II, IIM)-class singular configuration.

4.3. Singularity Classifications for Partially Non-Redundant Mechanisms

In this sub-section, we modify the general classification established with Theorem 31 and obtain classifications for two important special cases of redundant mechanisms. Herein, we are concerned with partially non-redundant mechanisms, i.e., mechanisms for which at least one of the two non-redundancy equalities, $n = n_O$ and $n = n_I$, holds. A mechanism is referred to as *kinematically non-redundant* when $n = n_O$ (and $n \le n_I$). Alternatively, a mechanism is *dynamically non-redundant* when $n = n_I$ (and $n \ge n_O$). Most redundant mechanisms appearing either in applications or in the literature belong to one of these groups, therefore the two classifications presented below are of interest.

When one of the non-redundancy conditions holds, some of the statements in Remark 29, which are generally not true for redundant mechanisms, can be proven. This rules out some singularity combinations and, as a result, two classifications, with 24 non-empty classes each, are obtained.

36. Proposition. *Let the mechanism be kinematically non-redundant, i.e.,* $n = n_O$.

(i) $\boldsymbol{q} \in \{\mathrm{RPM}\} \Rightarrow$

$$(\boldsymbol{q} \in \{\mathrm{II}\} \text{ and } \boldsymbol{q} \in \{\mathrm{IO}\}) \text{ or } \boldsymbol{q} \in \{\mathrm{IIM}\},$$

(ii) $\boldsymbol{q} \in \{\mathrm{II}\} \Rightarrow \boldsymbol{q} \in \{\mathrm{IO}\}$ or $\boldsymbol{q} \in \{\mathrm{RO}\}$.

Proof

(i) Let $\boldsymbol{q}$ be an RPM-type singularity but not an IIM-type singularity. Proposition 28 (v) implies that the configuration belongs to the II type. It remains to establish that $\boldsymbol{q}$ is an IO-type singularity as well. $\boldsymbol{q} \in \{\mathrm{RPM}\}$ implies $d_{IO} > 0$. Since there is no IIM-type singularity, we must have $n = n_q$. Then, we can write:

$$d_O \geq d_{IO} > 0 = n_q - n = n_q - n_O,$$

where the last equality uses the kinematic non-redundancy. Thus, it is established that $d_O > n_q - n_O$, which is equivalent to the presence of an IO-type singularity.

(ii) $\boldsymbol{q} \in \{\mathrm{II}\}$ implies $d_I > n_q - n$. We assume that there is no IO-type singularity, therefore, $d_O = n_q - n_O$. We need to show that $d_I - d_{IO} > 0$.

To establish this we will use the inequality sequence in the proof of Proposition 28 (ix). We notice that this sequence can be used to prove $d_I - d_{IO} > 0$, even when in the last inequality of the sequence the sign ">" is replaced with "≥", i.e., when the sequence is modified as follows:

$$d_I - d_{IO} > (n_q - n) - d_{IO} = n_q - n + n_O - n_O - d_{IO} = (n_q - n_O) - d_{IO} - (n - n_O) = d_O - d_{IO} - (n - n_O) \geq 0.$$

Therefore, to be able to use the above sequence we need only $d_O - d_{IO} \geq n - n_O$ (since the other equalities and inequalities in the sequence were already established in the proof of Proposition 28 (v)). However, this last inequality is implied by the kinematic non-redundancy, since

$$d_O - d_{IO} \geq 0 = n - n_O. \qquad \square$$

37. Proposition. *Let the mechanism be dynamically non-redundant, i.e.,* $n = n_I$.

(i) $\boldsymbol{q} \in \{\mathrm{IIM}\} \Rightarrow$

$$(\boldsymbol{q} \in \{\mathrm{RI}\} \text{ and } \boldsymbol{q} \in \{\mathrm{RO}\}) \text{ or } \boldsymbol{q} \in \{\mathrm{RPM}\},$$

(ii) $\boldsymbol{q} \in \{\mathrm{II}\} \Rightarrow \boldsymbol{q} \in \{\mathrm{IO}\}$ or $\boldsymbol{q} \in \{\mathrm{RO}\}$.

Proof

(i) Let q be an IIM-type but not an RPM-type singularity. From Proposition 28 (vi) it follows that the configuration belongs to the RI-type. It remains to establish that $\boldsymbol{q}$ is an RO-type singularity as well. The IIM-type singularity is characterized by $n < n_q$. When there is no RPM-type singularity, we have $d_{IO} = 0$. Then, it follows that:

$$d_I - d_{IO} = d_I = n_q - r_I > n - r_I = n_I - r_I \geq 0.$$

This yields $d_I > d_{IO}$, which is equivalent to the presence of an RO-type singularity.

(ii) Assuming that there is no IO-type singularity, we have $d_O = n_q - n_O$ and (from the given RI-type singularity) $d_O - d_{IO} > n - n_O$.

As in Proposition 36 (ii), to prove the statement we will use a variation of the inequality sequence in the proof of Proposition 28 (ix). We notice that this sequence can be used to prove $d_I - d_{IO} > 0$ even if in the first inequality of the sequence the sign ">" is replaced with "≥", namely:

$$d_I - d_{IO} \geq (n_q - n) - d_{IO} = n_q - n + n_O - n_O - d_{IO} = (n_q - n_O) - d_{IO} - (n - n_O) = d_O - d_{IO} - (n - n_O) > 0.$$

Therefore, to be able to use the above sequence we need only to establish the additional inequality, $d_I \geq n_q - n$. This, however, is implied by the dynamic non-redundancy, since

$$d_I = n_q - r_I \geq n_q - n_I = n_q - n. \qquad \square$$

38. Theorem. *Let* S *be an arbitrary combination of some of the six singularity types. There exists a kinematically non-redundant mechanism with a configuration,* $\boldsymbol{q}$, *which belongs to all types in* S *and to no other types, if and only if* S *is marked with* "Y" *or* "D" *in Table 2.*

Proof. Similarly to the proof of Theorem 31, the present proof has of two parts.

(i) To prove that the blank cells correspond to impossible configurations, we can use Propositions 28 and 36. The 22 configurations marked with blank cells in Table 2 are impossible for any mechanisms, including kinematically non-redundant mechanisms. The cells in Table 2 marked by "K" correspond to classes (RPM, II), (RPM, II, IIM) and (RO, RPM, II). Such singular configurations are disproved by Proposition 36.

(ii) We know that 21 of the combinations, marked with "Y" in Table 2, are possible. (Since non-redundant mechanisms are a special case of kinematically non-redundant mechanisms). The remaining three are (RI, IIM), (RI, IO, IIM) and (RI, RPM, IIM). The last two of these were already shown in Examples 32 and 34. In Example 23, the class (RI, IIM) was illustrated with a kinematically redundant mechanism. Below, we present an example with $n = n_O$.

39. Example. To prove the existence of (RI, IIM) singularities we consider a four-bar linkage, Figure 10. The output is defined as usual (the angular velocity of link *DC*), while dynamic redundancy is introduced by assuming that the joint at point *B* is active (in addition to joint *A*). Thus, we have $n_I = 2$, $n = n_O = 1$. In the flattened configuration, shown in the figure, the parameters determining the singularity types are $d_{IO} = 0$, $d_I = 0$, $d_O = 1$, $n_q = 2$. according to Table 1, the configuration belongs only to the types RI and IIM.

This completes the proof of Theorem 38. $\square$

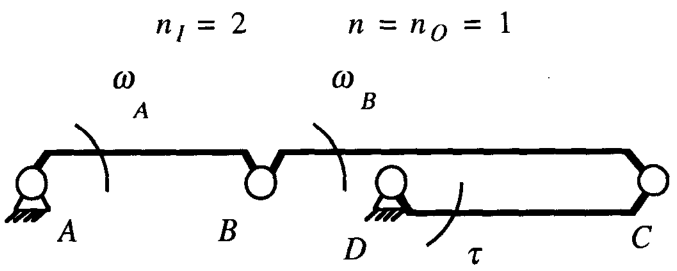

Figure 10. An (RI, IIM)-class singular configuration.

40. ***Theorem.*** *Let* S *be an arbitrary combination of some of the six singularity types. There exists a dynamically non-redundant mechanism with a configuration,* ***q****, which belongs to all types in* S *and to no other types, if and only if* S *is marked with "*Y*" or "*K*" in Table 2.*

Proof. Similarly to the previous Theorem 38, we need to prove that three combinations are impossible and establish that three other combinations are possible.

(i) The three classes, which are impossible for dynamically non-redundant mechanisms, but are possible for a general (redundant) mechanism, are (RI, IIM), (RI, IO, IIM) and (RI, RPM, IIM). Indeed, if such singularities were to exist for some mechanisms this would contradict Proposition 37.

(ii) The three classes, which occur for dynamically non-redundant mechanisms, but are impossible for non-redundant mechanisms, are (RPM, II), (RO, RPM, II) and (RPM, II, IIM). The existence of singularities from these classes is shown by Examples 27, 33 and 35 (Figures 4, 6 and 9), respectively.□

5. CONCLUSIONS

In this paper, a general framework for the singularity analysis of redundant and non-redundant mechanisms was developed. This was achieved by the generalization of the ideas introduced by Zlatanov et al. (1995) for non-redundant mechanisms. The six singularity types, were re-introduced with new, generalized definitions which remain relevant even when the mechanism is redundant. Using the motion-space model of instantaneous kinematics, the interdependence of the singularity types was examined. A comprehensive classification of the singular configurations of arbitrary mechanisms was obtained. It was shown that there are 27 different singularity classes, which can occur for various redundant mechanisms.

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